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## SECTIONS I. II. III.

WITH

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# NEWTON'S PRINCIPIA.

## SECTIONS I. II. III.

WITH

## NOTES AND ILLUSTRATIONS.

ALSO

A COLLECTION OF PROBLEMS

PRINCIPALLY INTENDED AS EXAMPLES OF NEWTON'S METHODS.



BY

PERCIVAL FROST, M.A.

LATE FELLOW OF ST JOHN'S COLLEGE;
MATHEMATICAL LECTURER OF JESUS COLLEGE.

*Principiis enim cognitis, multo facilius extrema intelligetis.*—CICERO.

Cambridge:
MACMILLAN AND CO.
1854

# PREFACE.

IN publishing the following work, the Author's principal intention has been to explain those difficulties which may be encountered on first reading the *Principia*, to illustrate the advantages of Newton's methods by shewing the extent to which they may be applied in the solution of problems, and to prepare the student for engaging in the study of the higher branches of mathematics by exhibiting in a geometrical form some of the processes employed in the Differential and Integral Calculus, and in the Analytical investigations in Dynamics.

The Author has endeavoured, in preparing the version of the first three Sections of the *Principia*, to adhere closely to the original form in the first Section and in the beginning of the second; and in the cases in which sentences have been interpolated or the form of the demonstration changed, such changes and interpolations have been marked by brackets.

In the second and third sections these indications of deviation from the original form have been discontinued.

In the first section in which the Lemmas are established, which are the basis of Newton's investigations, although it is generally advisable not to deviate from the original, yet in some cases his demonstrations are purposely expressed very concisely, and, in the fifth Lemma, he is contented with simply giving the enunciation, so that in these cases it is considered that no distinctness has been lost by the interpolations which have been made.

Throughout the Problems and Theorems which depend upon the sixth proposition, the variations have been replaced by equations. By this method of treating the subject, now commonly adopted in treatises professing to be versions of these Sections of the *Principia*, it is believed that clearer ideas of the meaning of each step of the demonstrations are obtained by the student.

The Author desires to make his acknowledgments of the great assistance which he has derived in the preparation of the Notes, from the study of Whewell's *Method of Limits*, from which the Articles 49 to 56 have been almost entirely taken: he has also made use of several editions of Newton, and especially of Carr's. He takes this opportunity of thanking several of his friends who have kindly given him their assistance in preparing his papers for the press, and in verifying the results of the Problems.

The principal portion of the Problems have been selected from the papers set in the examinations for the Mathematical Tripos and in the course of the College examination, especially in those of St John's College.

The Author has been influenced in this selection by his desire practically to bring before the student the great advantages which undoubtedly arise from a judicious use of geometrical methods.

It is only necessary to add, that care has been taken that no problems be introduced which are not capable of solution by methods given in the work.

CAMBRIDGE,
*October* 25, 1854.

# ERRATA.

Page 30. $1-\lambda^{2n}$ *for* 1.

" 180. XV. 2, the force $\propto \dfrac{\text{vertical ordinate}}{(\text{arc})^3}$.

# NEWTON'S FIRST BOOK

## CONCERNING THE MOTION OF BODIES.

---

## SECTION I.

### *On the Method of Prime and Ultimate Ratios.*

### LEMMA I.

*Quantities, and the ratios of quantities, which, in any finite time, tend constantly to equality, and which, before the end of that time, approach nearer to each other than by any assigned difference, become ultimately equal.*

If not, let them become ultimately unequal, and let their ultimate difference be *D*. Hence, [since, throughout the time, they tend constantly to equality,] they cannot approach nearer to each other than by the difference *D*, contrary to the hypothesis, [that they approach nearer than by *any* assigned difference. Therefore, they do not become ultimately unequal, that is, they become ultimately equal.]

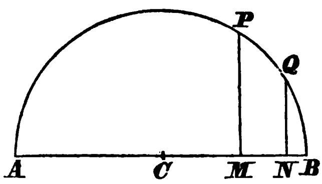
---

*Variable Quantities.*

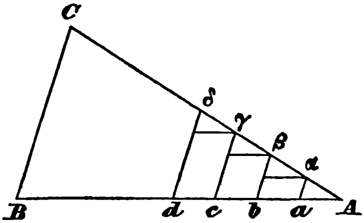
1. The *Quantities*, of which Newton treats in this Lemma, are variable magnitudes, described by a supposed law of construction, the variation of these magnitudes being due to the arbitrary progressive change of some element of the construction employed in the statement of the law.

When, in the progressive change of this element, it receives the last value which is assigned to it in any proposition, the hypothesis is said to arrive at its ultimate form.

Thus, if $ABP$ be a semicircle, $ACB$ its diameter, $BP$ any arc, $PM$ the ordinate perpendicular to $ACB$, as the arc $BP$ gradually diminishes, $AM$ is a variable magnitude, continually increasing, and $BP$ is the element of the construction, to the arbitrary change of which the variation of $AM$ is due; and if $BP$ may be made as small as we please, $AM$ approaches to $AB$, and the hypothesis approaches its ultimate form.




Again, if $ABC$ be a triangle, and $AB$ be divided into a number of equal portions, $Aa$, $ab$, $bc$,... and a series of parallelograms be inscribed upon those bases, whose sides $a\alpha$, $b\beta$, $c\gamma$,... are parallel to $BC$ and terminated in $AC$, the sum of the areas of the parallelograms will be a variable magnitude, defined by that construction, and changing in a progressive manner, if the number of parts into which $AB$ is divided is continually increased. In this case the number of parts is the variable element of the construction. In the ultimate form of the hypothesis, it will be shewn (Lemma II.) that the sum of the parallelograms is the area of the triangle, when the number is increased indefinitely.




2. The variation of a magnitude is *continuous*, when in the passage from *any* one value to *any* other, throughout its change, it receives every intermediate value, without becoming infinite. When this is not the case the variation is *discontinuous*.

According to the hypothesis in the last illustration, the number of parts into which $AB$ is divided being exact, the magnitude varies dis-

continuously, i. e. the sum of the areas does not pass through all the intermediate values between any two states of the progress.

If the hypothesis be changed, equal portions being set off commencing from $B$, and $Aa$ remaining over and above after $ba$, the last of the portions for which there is room, these equal portions could be made to diminish gradually, and the sum of the areas would vary continuously.

*Tendency to Equality.*

3. **Quantities are ultimately equal, when they are ultimately in a ratio of equality.**

4. **Quantities, which always remain *finite*, throughout the change of the hypothesis, by which they are described, tend continually to equality, when their difference continually diminishes.**

Thus, if $BQ$ be an arc, always half of $BP$, in fig. 1, and $QN$ the corresponding ordinate; as $BP$ continually diminishes, $AM$ and $AN$ remain finite, and their difference continually diminishing, they tend continually to equality.

5. **Quantities, which may become indefinitely small, as the hypothesis is indefinitely extended, tend continually to equality, when the ratio of their difference to either of them continually diminishes.**

To illustrate this tendency to equality, we observe that $BM$ and $BN$ have a difference, which tends continually to become $3BN$, the ratio of which to either is finite, so that, although both tend to become indefinitely small, as the hypothesis tends to its ultimate form, $BM$ and $BN$ do not satisfy the condition requisite for a tendency to equality.

Draw the chords $BP$, $BQ$, these are equal to the arcs when they are diminished indefinitely, as will be shewn.

$$\text{And } (\text{chord } BP)^2 = AB \,.\, BM,$$
$$(\text{chord } BQ)^2 = AB \,.\, BN;$$

$$\therefore BM : BN :: (\text{chord } BP)^2 : (\text{chord } BQ)^2,$$
$$:: (\text{arc } BP)^2 : (\text{arc } BQ)^2,$$
$$:: 4 : 1;$$
$$\therefore MN : BN :: 3 : 1.$$

*Observations on the Lemma.*

We will now proceed to examine the force of the other important terms employed in the statement of the first Lemma.

6. The expression "in any finite time" (tempore quovis finito), signifies what has been called the indefinite extension of the hypothesis, from some definite state to its ultimate form*.

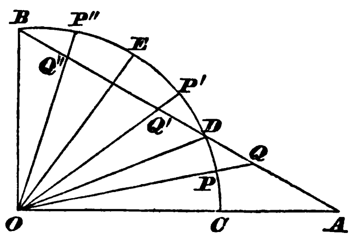
The law of the variation of the magnitudes under consideration is obtained by the examination of their construction while the element, to which the change is due, is at a finite distance from its final value, and the finite time is the supposed time occupied in the passage from this definite to the ultimate state.

In the first illustration (Art. 1), it denotes the progressive diminution of $BP$, from being a *finite* magnitude to the point of evanescence.

In the second, the progress from *any* finite number of equal portions to an indefinite number.

7. The expression, "which *constantly* tend" (quæ constanter tendunt), signifies that from the commencement of the *finite time* to the limit of the extension of the hypothesis, the differences continually diminish.

To illustrate this mode of expression, let $BC$ be a quadrant of a circle whose bounding radii are $OB$, $OC$, and let $BDA$ be a straight line cutting the arc $BDC$ and the radius $OC$ in $D$ and $A$, and let $OP$ be a radius revolving from $OC$ to $OB$, and cutting $BA$ in $Q$, $E$ the bisection of the arc $BD$, $OP$ and $OQ$ twice tend *to* equality, viz. from




* Vide Whewell's *Doctrine of Limits*.

$OC$ to $OD$ and from $OE$ to $OB$, and once *from* equality from $OD$ to $OE$; it is only from $OE$ to $OB$ that $OP''$ and $OQ'$ tend to equality *continually*, during the progress, and it is from such a position as $OE$ that the finite time must be considered to commence.

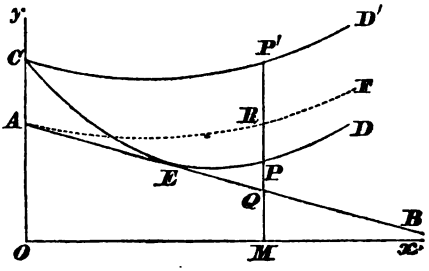
8. "Before the end of that time," (ante finem temporis,) implies that however small the given difference may be, a less difference than that difference is arrived at, while the distance from the ultimate state is finite, however near to the final state it may be necessary to proceed.

Thus, if, in the last figure, the angle $BOD$ be $60^0$, the radius one inch, and the given difference $\frac{35}{100000}$ or $\frac{18}{100000}$ of an inch, the difference is less, if the revolving radius be $2'$ or $1'$ from the ultimate position; and so on, however small the difference which is chosen.

9. In the proof of the Lemma, if the ultimate difference be $D$, the quantities cannot approach nearer than by that given difference; otherwise, they would, in one part of the progression, have been tending *from* equality in order to arrive ultimately at that difference, contrary to the statement of the proposition in the words, "ad æqualitatem constanter tendunt."

The nature of the proof, which is more difficult than may at first sight appear, can be illustrated as follows, by examining the effect of the omission of some of the points in the statement of the Lemma.

Draw $Oy$, $Ox$ at right angles, $AB$ any straight line meeting $Oy$ in $A$, $CED$ a curve touching $AB$ in $E$ and meeting $Oy$ in $C$, $CD'$ another touching a straight line parallel to $AB$ in $C$, $MQPP'$ a common ordinate.




As $OM$ diminishes until it becomes indefinitely small, $MQPP'$ moves up to $Oy$.

In both curves, the ordinates $MQ$ and $MP$ or $MP'$ have an ultimate difference $CA$, equal to $D$ suppose.

Omit the word "constanter," and the curve $CED$ is admissible in a representation of the approach of the quantities.

For the ordinates approach, before the end of the time, nearer than by any assignable difference, viz at $E$, but the condition of continual tendency to equality is not satisfied.

Omit the words "ante finem temporis, &c." and $CD'$ is sufficient.

For, in this case, they tend continually to equality, but before the end of the time they do not approach nearer than by any assignable difference, and they are ultimately unequal.

In the case of the dotted line $ARF$ touching $AB$ at $A$, all the conditions are satisfied. $QM$ and $RM$ tend continually to equality, and their difference may be made less than any given difference before $OM$ vanishes.

*Limit of a variable quantity.*

10. When a variable quantity tends continually to equality with a certain fixed quantity, and approaches nearer to this quantity than by any assignable difference, as the hypothesis determining its variation is approaching its ultimate form, this fixed quantity is called the *Limit of the variable quantity.*

The tests are,—that there should be a tendency to equality;—that this tendency should be continued from some finite condition;—and that the approach should, during the progression to the ultimate form, be nearer than by any assignable difference.

Thus, as is mentioned in the Scholium at the end of the section, the variable quantity does not become equal to, or surpass the limit, before the arrival at the ultimate form.

*Limiting ratio of variable quantities.*

11. If two quantities continually diminish or increase, and the ratio of these quantities tends continually to equality with a certain fixed ratio, and may be made to differ from that ratio by less than any assignable difference, as the hypothesis deter-

mining their variation is indefinitely extended; this fixed ratio is called the *limiting ratio* of the varying quantities.

*Ultimate ratio of vanishing quantities.*

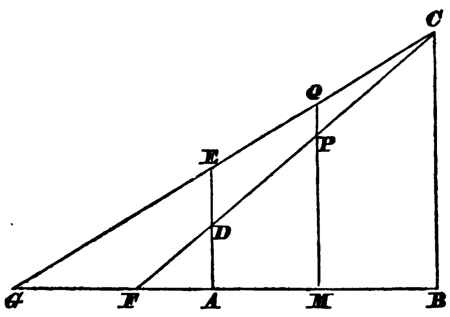
12. When the ultimate form of the hypothesis brings the quantities to a state of evanescence, they are called *vanishing quantities;* and the limiting ratio, or the limit of the ratio, is the *ultimate ratio of the vanishing quantities.*

The expression, "Vanishing quantities," does not imply that the quantities *are* indefinitely small while under examination, but only that they *will be* so in the ultimate form; which observation implies that *the ratio of the vanishing quantities* is not an equivalent expression with, *the ultimate ratio of the vanishing quantities*, the former being taken "ante finem temporis."

"Ultimæ rationes illæ quibuscum quantitates evanescunt, revera non sunt rationes quantitatum ultimarum." See Scholium, at the end of the section.

Thus,

Let $GC$, $FC$ be two straight lines intersecting $AB$ in $G$, $F$, $ADE$, $MPQ$, perpendicular to $AB$.




Let $\alpha$, $\beta$ be the areas $AMPD$, $AMQE$,

$$\alpha : \beta :: AD+MP : AE+MQ,$$

and let $MPQ$ be supposed to move up to $ADE$, then, in the ultimate

form of the hypothesis, $\alpha$ and $\beta$ vanish, and are called vanishing quantities from this circumstance.

Also, the *ultimate ratio* of the vanishing quantities is $AD : AE$.

Whereas, since $MP : MQ$ is not equal to $AD : AE$, the *ratio* of the vanishing quantities $AD+MP : AE+MQ$ is different from $AD : AE$.

*Prime Ratios.*

13. If the order of the change in the form of the hypothesis be reversed, or the varying quantities be tending from equality, having started into existence from the commencement of the time, the quantities are called *nascent quantities*; and the ratio with which they commence existence is called the *prime ratio* of the nascent quantities.

*Application of Lemma I. to the investigation of certain Limits.*

1. Limit of $\dfrac{1+x}{2-x}$, as $x$ gradually diminishes, and ultimately vanishes.

Since the difference between $\dfrac{1+x}{2-x}$ and $\dfrac{1}{2}$ is $\dfrac{3x}{2(2-x)}$, this difference continually diminishes as $x$ gradually diminishes, and, by diminishing $x$ sufficiently, may be made less than any assignable difference.

Hence, $\dfrac{1+x}{2-x}$ tends continually to equality with $\dfrac{1}{2}$, if we commence from some value of $x$ less than 2, and the difference may be made less than any assignable quantity *ante finem temporis*, therefore $\dfrac{1}{2}$ is the limit required.

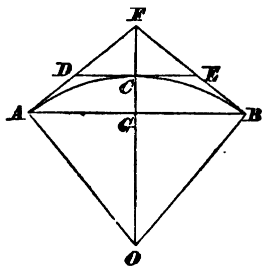
2. Limit of $\dfrac{2+x}{5+3x}$, when $x$ increases indefinitely.

Since the difference $\dfrac{2+x}{5+3x}-\dfrac{1}{3}=\dfrac{1}{3(5+3x)}$ which continually diminishes as $x$ increases, and may be made less than any assignable dif-

ference; therefore, as before, $\frac{1}{3}$ satisfies all the conditions of being a limit of $\frac{2+x}{5+3x}$.

3. Tangents are drawn to a circular arc, at its middle point, and at its extremities. Shew that the area of the triangle formed by the chord of the arc, and the two tangents at the extremities, is ultimately four times that of the triangle formed by the three tangents.

Let $C$ be the middle point of the arc, $AB$ the chord, $FA$, $FB$, $DCE$ the three tangents,




$$\triangle FDE : \triangle FAB :: FC^2 : FG^2.$$

Now $FC(FC+2CO) = FA^2$
$= FO \,.\, FG;$

$\therefore FC : FG :: FO : FC + 2CO;$

$\therefore$ since $FC$ vanishes in the limit,

$FC : FG :: CO : 2CO$ ultimately;

$\therefore FG = 2FC$ ultimately,

and $\triangle FDE : \triangle FAB :: 1 : 4$.

4. Limit of $\frac{x^m - 1}{x - 1}$, when $x$ differs from 1 by an indefinitely small quantity, $m$ being any number, rational or integral, positive or negative.

1st, where $m$ is a positive whole number

$$\frac{x^m - 1}{x - 1} = x^{m-1} + x^{m-2} + \ldots + x + 1,$$

which may be made to differ from $m$ by less than any assignable difference by taking $x$ sufficiently near to unity;

$$\therefore m \text{ is the limit of } \frac{x^m - 1}{x - 1}.$$

2ndly, Let $m = \frac{p - q}{r}$, $p$, $q$, and $r$ being positive whole numbers,

and let $x = y^r$;

$$\therefore \frac{x^m-1}{x-1}=\frac{y^{p-q}-1}{y^r-1}$$

$$=\frac{1}{y^q}\cdot\frac{y^p-y^q}{y^r-1}$$

$$=\frac{1}{y^q}\cdot\frac{y^p-1-(y^q-1)}{y^r-1}$$

$$=\frac{1}{y^q}\cdot\frac{\dfrac{y^p-1}{y-1}-\dfrac{y^q-1}{y-1}}{\dfrac{y^r-1}{y-1}}.$$

This may be made to differ from $\frac{p-q}{r}$ by a quantity less than any assignable quantity by taking $x$, and $\therefore y$, sufficiently near to unity;

$$\therefore m \text{ or } \frac{p-q}{r} \text{ is the limit required.}$$

When we divide the numerator and denominator by $y-1$, $y$ is not equal to 1, the time chosen being *ante finem temporis*, while the difference is finite: as in the directrix in the Scholium referred to above; "Cave intelligas quantitates magnitudine determinatas, sed cogita semper diminuendas sine limite."

5. Limit of $\dfrac{1^p+2^p+3^p+\ldots+n^p}{n^{p+1}}$, when $n$ is indefinitely increased.

Since this sum is the arithmetic mean of the $n$ fractions

$$\left(\frac{1}{n}\right)^p,\ \left(\frac{2}{n}\right)^p,\ \ldots\ldots\ \left(\frac{n}{n}\right)^p,$$

therefore, for all positive values of $p$, integral or fractional, it lies between

$$\left(\frac{1}{n}\right)^p \text{ and } \left(\frac{n}{n}\right)^p \text{ or } 1,$$

$\therefore$ its ultimate value lies between 0 and 1.

This being an important limit, we will investigate it first for the particular case in which $p$ is integral and positive, and then generally, when $p$ is any positive quantity.

Let $S_n = 1^p + 2^p + \ldots\ldots + n^p$.

Then $S_{n+1} = 1^p + 2^p + \ldots\ldots + n^p + (n+1)^p$;

$$\therefore\ S_{n+1} - S_n = (n+1)^p.$$

If therefore we assume that

$$S_n = An^{p+1} + Bn^p + \ldots\ldots + Ln + M,$$

$$\therefore\ S_{n+1} = A\overline{n+1}|^{p+1} + B\overline{n+1}|^p + \ldots\ldots + L(n+1) + M,$$

$$(n+1)^p = A(\overline{n+1}|^{p+1} - n^{p+1}) + B(\overline{n+1}|^p - n^p) + \ldots\ldots$$
$$+ \&c.$$
$$= A\{(p+1)n^p + (p+1).\frac{p}{2}n^{p-1} + \ldots\ldots\}$$
$$+ B(pn^{p-1} + p.\frac{p-1}{2}n^{p-2} + \ldots\ldots) + \ldots\ldots$$
$$+ \&c.$$

we obtain, by equating coefficients, $p$ equations for determining the $p$ constants $A, B, \ldots\ldots L$, &c. which reduce the equation to an identity.

The first of these equations is

$$1 = (p+1)A;$$

$$\therefore\ S_n = \frac{1}{p+1}.n^{p+1} + Bn^p + \ldots\ldots$$

$$\text{and } \frac{S_n}{n^{p+1}} = \frac{1}{p+1} + \frac{B}{n} + \frac{C}{n^2} + \ldots\ldots + \frac{M}{n^{p+1}},$$

the number of the terms following $\frac{1}{p+1}$ being finite.

Hence, if $n$ be increased, we may make the difference between

$$\frac{S_n}{n^{p+1}} \text{ and } \frac{1}{p+1}$$

diminish until it becomes less than any assignable quantity;

$$\therefore\ \frac{1}{p+1} \text{ is the limit required.}$$

Next, let $p$ be any positive quantity, and let $l$ be the limit of

$$\frac{1^p + 2^p + \ldots\ldots + n^p}{n^{p+1}},$$

$$\therefore \; 1^p + 2^p + \dots\dots + n^p = ln^{p+1} + Bn^\beta + Cn^\gamma + \dots\dots$$

where $p+1$, $\beta$, $\gamma$ ...... are in descending order, and $B$, $C$, ...... constant,

and $\dfrac{Bn^\beta + Cn^\gamma + \dots\dots}{n^{p+1}}$ vanishes, when $n$ is made infinitely large.

$$\therefore \; 1^p + 2^p + \dots\dots + \overline{n+1}|^p = l\,\overline{n+1}|^{p+1} + B\overline{n+1}|^\beta + \dots$$

$$\therefore \; \overline{n+1}|^p = l(\overline{n+1}|^{p+1} - n^{p+1}) + B(\overline{n+1}|^\beta - n^\beta) + \dots\dots$$

$$\therefore \; \left(1+\frac{1}{n}\right)^p = l.\frac{\left(1+\frac{1}{n}\right)^{p+1} - 1}{1+\frac{1}{n}-1} + \frac{Bn}{n^{p+1}}.\frac{\left(1+\frac{1}{n}\right)^\beta - 1}{1+\frac{1}{n}-1}$$

$$+ \dots\dots\dots\dots\dots\dots\dots\dots$$

$\therefore$ observing that, when $n$ is increased indefinitely,

$$\frac{\left(1+\frac{1}{n}\right)^q - 1}{1+\frac{1}{n}-1} = q,$$

$$\therefore \; 1 = (p+1)\,l + \text{limit of } \frac{\beta(1+\epsilon)Bn^\beta + \gamma(1+\epsilon')Cn^\gamma + \dots\dots}{n^{p+1}};$$

where $\epsilon$, $\epsilon'$,... vanish ultimately. If now $\epsilon_1$ be the greatest of the quantities $\epsilon$, $\epsilon'$,...

$$\frac{\beta(1+\epsilon)Bn^\beta + \dots\dots}{n^{p+1}} \text{ is less than } (1+\epsilon_1)\,\beta \times \frac{Bn^\beta + \frac{\gamma}{\beta}Cn^\gamma + \dots\dots}{n^{p+1}};$$

$\therefore$ since $\dfrac{\gamma}{\beta}$, $\dfrac{\delta}{\beta}$ ...... are each $< 1$;

$$1 = (p+1)\,l \text{ ultimately};$$

$$\therefore \; \frac{1}{p+1} \text{ is the limit required.}$$

Cor. $\frac{1}{p+1}$ is evidently the limit of the sum

$$\frac{1^p + 2^p + \ldots\ldots + \overline{n-1}|^p}{n^{p+1}}.$$

Since $\frac{n^p}{n^{p+1}}$ vanishes in the limit.

6. **Limit of $(\cos\theta)^{\frac{a^2}{\theta^2}}$, when $\theta$ is indefinitely diminished.**

Let $u = (\cos\theta)^{\frac{a^2}{\theta^2}}$;

$$\therefore \log(u) = \frac{a^2}{\theta^2}\log(\cos\theta) = \frac{a^2}{2\theta^2}\log(1 - \sin^2\theta)$$

$$= \frac{a^2}{2}\cdot\frac{\sin^2\theta}{\theta^2}\cdot\frac{1}{\sin^2\theta}\log(1 - \sin^2\theta).$$

Now $\frac{\sin\theta}{\theta} = 1$ ultimately, (Lemma VII.)

$$\text{and } \frac{1}{\sin^2\theta}\log(1 - \sin^2\theta)$$

$$= -(1 + \frac{1}{2}\sin^2\theta + \frac{1}{3}\sin^4\theta + \ldots\ldots)$$

and the series in this bracket is greater than 1, and less than

$$1 + \sin^2\theta + \sin^4\theta + \ldots\ldots \text{ or } \frac{1}{1 - \sin^2\theta} = \sec^2\theta,$$

and the difference between 1 and $\sec^2\theta = \tan^2\theta$ vanishes in the limit;

$$\therefore \frac{1}{\sin^2\theta}\log(1 - \sin^2\theta) = 1 \text{ ultimately};$$

$$\therefore \log(u) = -\frac{a^2}{2} \text{ ultimately};$$

$$\therefore u = e^{-\frac{a^2}{2}} \text{ which is the limit required.}$$

## I.

1. Are the limits of the ratios $y^2 : x$ equal in any of the three equations,

$$(1)\ y^2 = ax, \quad (2)\ y^2 = ax - b^2, \quad (3)\ y^2 = av - x^2,$$

when $x$ is indefinitely diminished?

2. Find the limit of $\dfrac{x+3}{1+3x}$,

(1) when $x$ is indefinitely diminished,

(2) when $x$ is indefinitely increased.

3. Find the ultimate ratio of the vanishing quantities $ax + bx^2$, $bx + ax^2$, when $x$ is made indefinitely small.

4. $BAC$, $bAc$ are two triangles, in which $AB$, $Ab$ and $AC$, $Ac$ are coincident in direction, and $BC$, $bc$ intersect in $P$; prove that if the areas of the triangles are equal, as $B$, $C$ and $b$, $c$ approach, each to each, $P$ is ultimately in the point of bisection of $BC$.

5. If the angle $A$ in the last construction be a right angle, and the hypothenuses $BC$, $bc$ be equal, shew that the ultimate ratio of

$$Ac - AC : AB - Ab \text{ is } AB : AC,$$

and that $PB : PC :: AC^3 : AB^2$.

6. If in the right-angled triangles $ABC$, $Abc$ the perimeters be equal, shew that the ultimate ratio of the vanishing quantities

$$Bb : Cc \text{ is } AC + BC : AB + BC.$$

7. Also shew that the ultimate ratio of the areas

$BPb : CPc$ is $(BC + AC)(BC - AB) : (BC - AC)(BC + AB)$.

8. Tangents are drawn to a circular arc at its middle point, and at its extremities, and the three chords are drawn. Find the limit of the ratio of the triangles contained by the three tangents and the three chords when the arc is indefinitely diminished; shew that it is 1 : 2.

9. In the last construction shew that one of the triangles contained by two tangents and a chord is eight times either of the two other triangles, when the arc is indefinitely diminished.

10. $APQ$ is a parabola, $PM$, $QN$ ordinates to the axis $AMN$, with centres $M$ and $N$ and radii $PM$, $QN$ two circles are drawn, shew that when $N$ approaches indefinitely near to $M$, the distance of the point of intersection of the two circles from $PM$ is ultimately equal to the semi-latus rectum.

11. $APQ$, $ABC$ are two straight lines which are intersected by two fixed lines $BP$, $CQ$, prove that as $APQ$ moves up to $ABC$, $PC$ and $QB$ intersect in a point whose ultimate position divides $BC$ in the ratio of $AB : AC$.

12. $ABC$, $APQ$ are drawn to cut a circle from an external point $A$, $BU$, $CT$ are tangents at $B$ and $C$ to the circle, meeting $APQ$ in $U$, $T$; shew that the ultimate ratio of $PU : QT$ is $AB^2 : AC^2$.

13. $AB$, $AC$ are two right lines; $AC_1, AC_2, AC_3, \ldots AC_n$ are set off on $AC$ in the proportion of $1, 2, 3, \ldots n$, $S$ is a fixed point, and the right lines $C_1S, C_2S, \ldots C_nS$ cut $AB$ in $D_1D_2\ldots D_n$.

Shew that if $AC_1$ be very large compared with the distance of $S$ from $AB$, the limiting ratio of

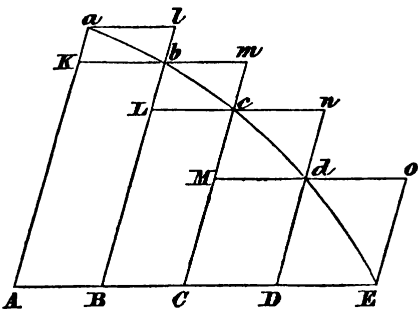
$$D_{n-1}D_n : D_nD_{n+1} \text{ is } n+1 : n-1.$$

14. Find the limit of $\left(1+\frac{1}{n}\right)^n$ when $n$ is indefinitely increased.

15. Find the limit of $\frac{1}{n} l_e (1+n)$ when $n$ is indefinitely diminished.

## LEMMA II.

*If, in any figure* AacE, *bounded by the straight lines* Aa, AE *and the curve* acE, *any number of parallelograms* Ab, Bc, Cd, *&c. be inscribed, upon equal bases* AB, BC, CD, *&c., and having sides* Bb, Cc, Dd, *&c. parallel to the side* Aa *of the figure; and the parallelograms* aKbl, bLcm, cMdn, *&c. be completed; then, if the breadth of these parallelograms be diminished, and the number increased indefinitely, the ultimate ratios which the inscribed figure* AKbLcMdD, *the circumscribed figures* AalbmcndoE, *and the curvilinear figures* AabcdE, *have to one another, are ratios of equality.*

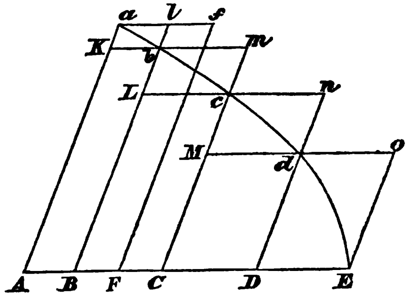



For the difference of the inscribed and circumscribed figures is the sum of the parallelograms $Kl$, $Lm$, $Mn$,

*Do*, that is, (since the bases of all are equal) a parallelogram whose base is *Kb*, that of one of them, and altitude the sum of their altitudes, that is, the parallelogram *ABla*. But this parallelogram, since its breadth is diminished indefinitely, [as the number of parallelograms is increased indefinitely,] becomes less than any assignable parallelogram, therefore (by Lemma I), the inscribed and circumscribed figures, and, *a fortiori*, the curvilinear figure, which is intermediate, become ultimately equal.

## LEMMA III.

*The same ultimate ratios are also ratios of equality, when the breadths of the parallelograms* AB, BC, CD,.... *are unequal, and all are diminished indefinitely.*




For, let *AF* be equal to the greatest breadth, and the parallelogram *FAaf* be completed. This parallelogram will be greater than the difference between the inscribed and circumscribed figures. But, when its breadth is diminished indefinitely, it will become less than any assignable parallelogram. [Therefore, *a fortiori*, the difference between the inscribed and circumscribed figures will become less than assignable areas. Hence, by Lemma I, the ultimate ratios of the inscribed and

circumscribed and the curvilinear figure (which is intermediate) will be ratios of equality.]

Cor. 1. Hence the ultimate sum of the vanishing parallelograms coincides [as to area] with the curvilinear figure.

Cor. 2. And, *a fortiori*, the rectilinear figure which is bounded by the chords of the vanishing arcs $ab$, $bc$, $cd$, &c. ultimately coincides [as to area] with the curvilinear figure.

Cor. 3. As also the rectilinear circumscribed figure, which is bounded by the tangents of the same arcs.

Cor. 4. And these ultimate figures, with respect to their perimeters $acE$, are not rectilinear, but curvilinear limits of rectilinear figures.

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14. The Propositions concerning limits of Quantities and their ratios state, (1) the hypothesis by which the quantities are defined, (2) the manner in which the hypothesis approaches its ultimate form, and (3) the ultimate property when the hypothesis is thus indefinitely extended.

The strength of the proofs lies in the examination of the quantities, while the hypothesis is in a finite state, before arrival at the ultimate form, and the deduction of properties by which the relations of the quantities can be pursued accurately to the ultimate state.

In the statement of the Lemmas II, and III, the hypothetical constructions give the manner of describing the parallelograms; the extension of the hypothesis towards its ultimate form is the continual increase of the number of parallelograms *in infinitum;* the limiting property, is the equality of the ratio of the sums of the parallelograms and the curvilinear area.

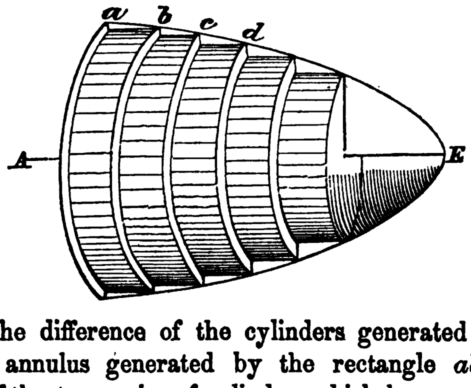
In the proof of the Lemmas, the continual decrease of the parallelo-

grams $Al$ or $Af$, shews that the conditions of ultimate equality of two quantities are all satisfied, *viz.*, that the sums of the two series of parallelograms tend continually to equality, and that they approach nearer to each other than by any assignable difference, "ante finem temporis," while the number of the parallelograms remains finite.

*Volumes of Revolution.*

15. In a manner exactly similar to Lemma II, it may be shewn, that, if $Aa$ be perpendicular to $AE$, and the whole figure revolve round $AE$ as an axis, the ultimate ratios which the sums of the volumes of the cylinders, generated respectively by the rectangles $Ab$, $Bc$,... and $aB$, $bC$,... and the volume of revolution generated by the curvilinear area $AEa$ have to each other, are ratios of equality.

The figure represents the cylinders generated by the inscribed rectangles.



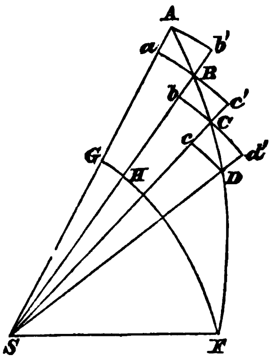

Thus, the difference of the cylinders generated by $Ab$ and $aB$ is the annulus generated by the rectangle $ab$, and the difference of the two series of cylinders which have each the same height, $AB = BC = \ldots$ is the sum of such annuli, and is easily seen to be the cylinder generated by $aB$, which, since the height continually diminishes, may be made less than any assignable volume, and the conditions, that the two series may have the same limit, are satisfied, and hence also the volume of revolution,

which is greater than one sum and less than the other, is ultimately in a ratio of equality to either sum.

The same argument applies, if the revolution be only through a certain angle instead of being complete: in which case the cylinders are replaced by sectors of cylindrical volumes.

*Sectorial Areas.*

16. The Lemmas may be extended to sectorial areas. Thus, if $SABCF$ be a sectorial area, and the angle $ASF$ be divided into equal portions $ASB$, $BSC$,...... and the circular arcs $Ab'$, $aBc'$, $bCd'$,...... be drawn with center $S$; then, since the difference of the two series of circular sectors is the sum of the areas $ab'$, $bc'$,...... it is equal to the difference of the greatest and least of the sectors, viz. $AGHb'$ and the two areas $SAb'Bc'$......, and $SaBbC$......, tend continually to equality as the number of angles is increased, and their magnitudes diminished, and the ratios which these areas have to each other and to the area $SABF$ are ultimately ratios of equality.




Similarly for Lemma III.

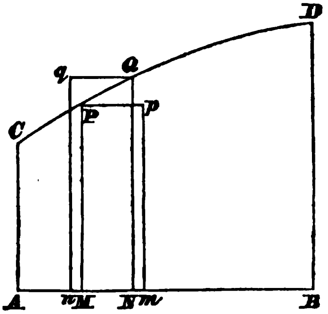
*Surfaces of Revolution.*

17. The following proposition is the extension of the principles of the Lemmas to the determination of a method for finding the area of a surface of a solid of revolution.

$CD$ is a plane curve which generates a surface of revolution by its revolution round $AB$ a line in its plane.

$CD$ is divided into portions of which $PQ$ is one, $PM$, $QN$

perpendicular to $AB$. $Pp$, $Qq$ are drawn parallel to $AB$, and each equal to $PQ$ in length, $pm$, $qn$ perpendicular to $AB$. The surface generated by $CD$ shall be the limit of the sum of the cylindrical surfaces generated by such portions as $Pp$ or $Qq$. For, the cylindrical surfaces generated by $Pp$ and $Qq$ are one less, and the other greater than that generated by $PQ$.




But these surfaces are respectively $2\pi PM \cdot Pp$ and $2\pi\, QN \cdot Qq$, and their difference is

$$2\pi\ (QN - PM)\ PQ,$$

the ratio of this difference to the surfaces themselves is

$$QN - PM : PM, \text{ or } QN,$$

which ratio is ultimately less than any given ratio.

Hence the sums of the surfaces generated by the lines corresponding to $Pp$ and $Qq$ have the ratio of their difference to either sum in a ratio less than the greatest ratio $QN - PM : PM$ which may be made less than any finite ratio. Therefore the sums of the cylindrical surfaces, and the curved surface, which is intermediate in magnitude, are ultimately in a ratio of equality.

*General Extension.*

18. If any magnitude $A$ be divided into a series of magnitudes $A_1 A_2 \ldots\ldots A_n$ and two series of quantities $a_1 a_2 \ldots\ldots a_n$ and $b_1 b_2 \ldots\ldots b_n$ can be found such that

$$a_1 > A_1 > b_1,$$
$$a_2 > A_2 > b_2,$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$a_n > A_n > b_n,$$

and $a_1 - b_1 : a_1$, $a_2 - b_2 : a_2 \ldots\ldots$ are ratios, each of which be-

comes less than any finite ratio, when the number is increased; then $a_1 + a_2 + \ldots\ldots + a_n$, $b_1 + b_2 + \ldots\ldots + b_n$ and $A$ are ultimately in a ratio of equality. For, let $l : 1$ be equal to the greatest of the ratios $a_1 - b_1 : a_1$, &c.

$$\therefore\ a_1 - b_1 + a_2 - b_2 + \ldots\ldots : a_1 + a_2 + \ldots\ldots$$

is a ratio less than $l : 1$, and may therefore be made less than any assignable ratio by increasing the number. Therefore the two series $a_1 + a_2 + \ldots\ldots$ and $b_1 + b_2 + \ldots\ldots$ tend continually to equality, and the difference may be made before the end of the time less than any assignable magnitude; therefore the three magnitudes are ultimately in a ratio of equality.

19. Cor. 1. "Omni ex parte" has not been adopted from the text of Newton, because it requires limitation, for the perimeters do not coincide with the perimeter of the curvilinear area.

In the figure for Lemma II, the perimeter of the inscribed series of parallelograms is

$$AK + Kb + bL + Lc + \ldots\ldots + EA = 2AK + 2AE,$$

and the limit of this perimeter is $2Aa + 2AE$.

The perimeter of the other series of parallelograms being also $2Aa + 2AE$ is constant throughout the change, and has properly no limit.

20. Cor. 2. The perimeter of the figure bounded by the chords $ab$, $bc$, ...... ultimately coincides with that of the curvilinear figure. This coincidence is discussed under Lemma V.

21. Cor. 3. The same is true for the figure formed by the tangents.

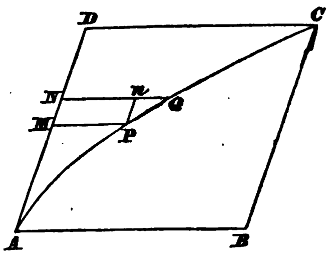
22. Cor. 4. Instead of "propterea" it would be advisable to *state*, as in Whewell's *Doctrine of Limits*, that, if a *finite*

portion of a curve be taken, and if many points be taken in the curve, so as to form a polygon, the sides of which are chords, taken in order, of portions of the curve, and if the number of those points be increased indefinitely, the curve is the limit of the polygon.

*Application to the determination of certain areas, volumes, &c.*

1. Area of a parabola bounded by a diameter and an ordinate.

Let $AB$, $BC$ be the bounding abscissa and ordinate. Complete the parallelogram $ABCD$.




Let $AD$ be divided into $n$ equal portions, of which suppose $AM$ to contain $r$ and $MN$ to be the $(r+1)^{\text{th}}$, draw $MP$, $NQ$ parallel to $AB$, meeting the curve in $P$, $Q$, and $Pn$ parallel to $MN$; the curvilinear area $ACD$ is the limit of the sum of the series of parallelograms constructed, as $PN$, on the portions corresponding to $MN$.

By the properties of the parabola,

$$PM : CD :: AM^2 : AD^2$$

$$:: r^2 : n^2,$$

$$\text{and } MN : AD :: 1 : n;$$

$$\therefore PM.MN : AD.CD :: r^2 : n^3;$$

$$\therefore \text{parallelogram } PN : \text{parallelogram } ABCD,$$

$$:: r^2 : n^3;$$

$$\therefore \text{parallelogram } PN = \frac{r^2}{n^3} \times \text{parallelogram } ABCD;$$

$$\therefore \text{the sum of the series of parallelograms}$$

$$= \frac{1^2 + 2^2 + \ldots\ldots + \overline{n-1}|^2}{n^3} \times \text{parallelogram } ABCD,$$

$$\text{and } \frac{1^2+2^2+\dots\dots+\overline{n-1}|^2}{n^3}=\frac{1}{3},$$

when the number is increased indefinitely,

$\therefore$ proceeding to the ultimate form of the hypothesis,

the curvilinear area $ACD=\frac{1}{3}$ of the parallelogram $ABCD$.

$\therefore$ the area $ABC=\frac{2}{3}$ of the parallelogram $ABCD$.

Cor. 1. If we had inscribed the series of parallelograms in $ABC$, $AB$ being divided into $n$ portions, we should have arrived at the result

$$\frac{1^{\frac{1}{2}}+2^{\frac{1}{2}}+\dots\dots+\overline{n-1}|^{\frac{1}{2}}}{n^{\frac{3}{2}}},$$

for the ratio of the series to the parallelogram $ABCD$, which might thus have been shewn to be ultimately $\frac{2}{3}$.

Cor. 2. If $BC$ had been divided into $n$ equal portions, the parallelogram corresponding to $PN$ would have been

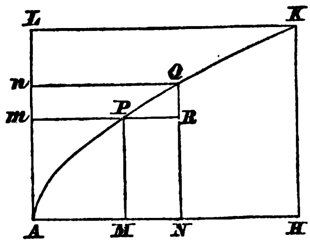
$$\frac{n^2-r^2}{n^3}\times\text{parallelogram } ABCD,$$

and the ratio of the area $ABC$ to the parallelogram $ABCD$, the limit of

$$\frac{n^2-1^2+n^2-r^2+\dots\dots+n^2-\overline{n-1}|^2}{n^3},$$

$$\text{of } 1-\frac{1^2+2^2+\dots\dots+\overline{n-1}|^2}{n^3}=1-\frac{1}{3}=\frac{2}{3}.$$

## 2. Volume of a paraboloid.




Let $AKH$ be the area of a parabola cut off by the axis $AH$ and an ordinate $HK$, which by its revolution round the axis generates a paraboloid.

Let $AH$ be divided into $n$ equal portions, and $PRNM$ the parallelogram on $MN$ be the $\overline{r+1}|^{\text{th}}$;

$$PM^2 : HK^2 :: AM : AH,$$
$$:: r : n,$$
$$MN : AH :: 1 : n;$$
$$\therefore\ PM^2.MN : HK^2.AH :: r : n^2.$$

Cylinder generated by $PN$ : cylinder by $AHKL$ :: $r : n^2$.

Cylinder generated by $PN = \frac{r}{n^2} \times$ cylinder by $AHKL$;

$\therefore$ the sum of the cylinders inscribed $= \dfrac{1+2+\ldots\ldots+(n-1)}{n^2}$

$\times$ circumscribed cylinder,

and when $n$ is indefinitely increased,

$$\frac{1+2+\ldots\ldots+\overline{n-1}]}{n^2} = \frac{1}{2} \text{ ultimately},$$

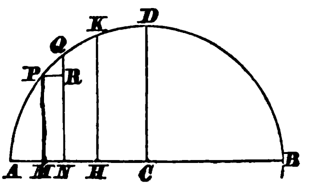
and the paraboloid is the limit of the series of inscribed cylinders;

$\therefore$ the volume of the paraboloid is

$\frac{1}{2} \times$ the cylinder on the same base and of the same altitude.

3. Volume of a spherical segment.

Let $AHK$ generate by its revolution round the diameter $AB$, the spherical segment whose height is $AH$.




Divide $AH$, as before,

$\therefore\ AM = \frac{r}{n} AH,$

and $PM^2 = AM.MB$

$= AM.(AB - AM)$

$= \frac{r}{n} AH.AB - \frac{r^2}{n^2} AH^2.$

Volume of cylinder generated by $PN$

$$= \pi PM^2.MN = \pi.\frac{AH}{n}.PM^2$$

$$= \pi AH^2.\left(\frac{r}{n^2} AB - \frac{r^2}{n^3} AH\right),$$

whence, as before, the limit of the sum

$$=\pi AH^2\left(\frac{AB}{2}-\frac{AH}{3}\right),$$

which is the volume proposed.

COR. 1. If $AH=\frac{1}{2}AB=AC$, the segment is a hemisphere whose volume

$$=\pi AC^2\left(AC-\frac{AC}{3}\right)$$

$$=\frac{2\pi AC^3}{3}$$

$=\frac{2}{3}\times$ cylinder on the same base and of the same altitude.

COR. 2. If $AH=2AC$,
the volume of the whole sphere

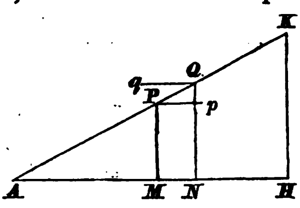
$$=4\pi AC^2\left(AC-\frac{2AC}{3}\right)$$

$$=\frac{4\pi AC^3}{3}.$$

### 4. Area of the surface of a right cone.

As an illustration of the method of finding surfaces given above, suppose $AHK$ a right-angled triangle, which revolves round $AH$, a side containing the right angle, $AK$ generates a conical surface.

Let $MN$ be the $\overline{r+1}|^{\text{th}}$ portion of $AH$, after division into $n$ equal portions, $MP$, $NQ$ ordinates parallel to $HK$, $Pp$, $Qq$ each equal to $PQ$ and parallel to $AH$.




The areas generated by $Pp$ and $Qq$ respectively are

$2\pi.PM.Pp$, and $2\pi.QN.Qq$,

and $PM : HK :: AM : AH :: r : n$,

$$QN : HK :: AN : AH :: (r+1) : n,$$

$$PQ : AK :: MN : AH :: 1 : n;$$

$\therefore$ the areas are $\frac{r}{n^2}.2\pi HK.AK$, and $\frac{r+1}{n^2}2\pi HK.AK$ respectively; and the conical surface is intermediate in magnitude to

$$2\pi.AK.HK \times \frac{1+2+\ldots\ldots+(n-1)}{n^2},$$

$$\text{and } 2\pi.AK.HK \times \frac{1+2+\ldots\ldots+n}{n^2};$$

each of which have for their limit $\pi AK.HK$ which is therefore the area of the conical surface.

Although not related to the method of limits, the reader may notice the following method of obtaining the conical surface by development.

If a circular sector $KAK'$ be cut out and the bounding radii $AK$, $AK'$ placed in contact so that the boundary $KLK'$ forms a circle.

The figure is conical, and $AK$ is the slant side, $HK$ in the last figure is the radius of the circular base whose length is $KLK'$.

The area of the conical surface : $\pi AK^2$

$$:: \text{sector } KLK' : \text{complete circle } KLK'K$$

$$:: 2\pi HK : 2\pi AK$$

$$:: \pi HK.AK : \pi AK^2;$$

$\therefore$ $\pi HK.AK$ is the area of the conical surface.

5. **Mass of a rod whose density varies as the $m^{\text{th}}$ power of the distance from the extremity.**

Let $AB$ be the rod, and let $MN$ be the $\overline{r+1}|^{\text{th}}$ portion, when its length is divided into $n$ equal parts.

$\rho . AM^m$ the density at $M$, or the quantity of matter contained in an unit of length of the rod supposed of the same substance as the rod at the point $M$.

The quantity of matter in $MN$ is intermediate between

$$\rho . AM^m . MN, \text{ and } \rho AN^m . MN,$$

the ratio of the difference of these to either of them being less than any assignable ratio when $n$ is indefinitely increased;

$$\text{and since } AM = \frac{r}{n} AB,$$

$$MN = \frac{1}{n} AB;$$

$\therefore$ the mass of the whole rod is the limit of

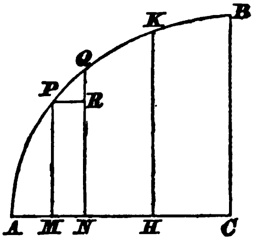
$$\rho . \frac{1^m + 2^m + \ldots\ldots + \overline{n-1}|^m}{n^{m+1}} AB^{m+1}$$

$$= \frac{1}{m+1} \times \rho . AB^{m+1}$$

$= \dfrac{1}{m+1}$ of the mass of a rod of uniform density equal to that of the rod $AB$ at $B$.

6. Center of gravity of the volume of a hemisphere.

Let $CAB$ be a quadrant which by its revolution round the radius $CA$ generates the hemisphere.




Let $MR$ be the rectangle which generates the $\overline{r+1}|^{\text{th}}$ inscribed cylinder;

$$CM = \frac{r+1}{n} \times CA;$$

$$MN = \frac{1}{n} \times CA.$$

The mass of the cylinder generated by $MR$ is proportional to

$$\pi PM^2 . MN$$

$$= \pi (CA^2 - CM^2) MN$$

$$=\left\{1-\frac{(r+1)^2}{n^2}\right\}\pi CA^2 . \frac{CA}{n}$$

$$=\frac{\pi CA^3}{n}-\frac{(r+1)^2}{n^3}\pi CA^3;$$

∴ the mass of the series of inscribed cylinders is proportional to

$$\pi CA^3-\frac{1^2+2^2+\ldots\ldots+n^2}{n^3}\pi CA^3;$$

∴ the mass of the hemisphere is proportional to

$$\pi CA^3-\frac{1}{3}\pi CA^3=\frac{2}{3}\pi CA^3.$$

Again, the moment of the mass of the cylinder generated by $MR$ with respect to the base of the hemisphere is proportional to

$$\pi QN^2 . MN . \frac{CM+CN}{2},$$

which differs from $\pi QN^2 . MN . CM$ by a quantity which vanishes compared with it, and is therefore ultimately proportional to

$$\left(\frac{r+1}{n^2}-\frac{\overline{r+1}|^3}{n^4}\right)\pi CA^4;$$

∴ the moment of the hemisphere is proportional to

$$\left(\frac{1}{2}-\frac{1}{4}\right)\pi CA^4, \text{ or } \frac{1}{4}\pi CA^4;$$

∴ the distance of the center of gravity of the volume of the hemisphere from $C$

$$=\frac{\text{moment with respect to the base}}{\text{mass}}$$

$$=\frac{\frac{1}{4}\pi CA^4}{\frac{2}{3}\pi CA^3}=\frac{3}{8} . CA.$$

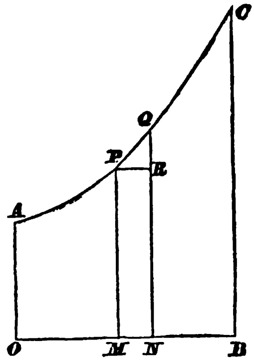
7. **Area of an equiangular spiral, between bounding radii $SA$, $SL$.**

pendicular to $AD$, and $FC$ is the greatest value of $BE$, and $BE : FC :: \sin\left(\frac{AB}{AD}\cdot\pi\right) : 1$. Shew that the area $ABE$ varies as $HG$, where $GK$ is the ordinate equal to $BE$ of the circle $CH$, whose center is $F$, and radius $FC$.

13. In the curve of the last problem, shew that the ratio of the area $ACD$ to the triangle, whose sides are $AD$, and the tangents $AT$, $DT$ at the extremities, is $8 : \pi^2$.

14. In the curve $APC$, the area contained between the curve and the abscissa $OB$ and rectangular ordinate $BC$ is $AO\,(BC - AO)$, the relation between any ordinate $PM$ and abscissa $OM$ being

$$OM : OA :: \log\frac{PM}{OA} : 1.$$




15. Shew that the center of gravity of a paraboloid of revolution is distant from the vertex two-thirds of the length of the axis.

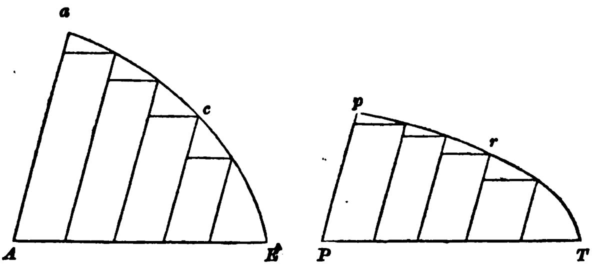
16. Shew that the abscissa and ordinate of the center of gravity of a parabolic area, contained between a diameter $AB$ and ordinate $BC$, are $\frac{3}{5}AB$ and $\frac{3}{8}BC$, respectively.

17. The limiting ratio of a hyperboloid of revolution, whose axis is the transverse axis, to the circumscribing cylinders, is $1 : 2$, when the altitude is indefinitely diminished, and $1 : 3$, when it is indefinitely increased.

18. The volume of a spheroid is one-third of the circumscribing cylinder.

## LEMMA IV.

*If in two figures* AacE, PprT, *there be inscribed (as in Lemmas* II, III) *two series of parallelograms, the number in each series being the same, and if, when the breadths are diminished indefinitely, the ultimate ratios of the parallelograms in one figure to the parallelograms in the other be the same, each to each; then, the two figures* AacE, PprT *are to one another in that same ratio.*




For, as the parallelograms are each to each, [they being ultimately in the same ratio,] so, componendo, is the sum of all to the sum of all, and so the figure *AacE* to the figure *PprT*, for, by Lemma III, the former figure is to the former sum, and the latter figure to the latter sum in a ratio of equality. Q.E.D.

Cor. Hence, if two quantities of any kind whatever, be divided into any, the same, number of parts; and those parts, when their number is increased, and magnitude diminished indefinitely, assume the same given ratio each to each, viz. the first to the first, the second to the second, and so on in order, the whole quantities will be to one another in the same given ratio. For, if, in the figures of this Lemma, the parallelograms be taken each to each in the same ratio as the parts, the sums of the parts will be always as the sums of the parallelograms:

and, therefore, when the number of the parts and parallelograms is increased, and their magnitude diminished indefinitely, the two quantities will be in the ultimate ratio of parallelogram to parallelogram, that is, (by hypothesis) in the ultimate ratio of part to part.

---

23. The general proposition contained in the Corollary may be proved in the following manner:

Let $A$, $B$ be two quantities of any kind, which can be divided into the same number $n$ of parts, viz. $a_1, a_2, a_3 \ldots\ldots a_n$, and $b_1, b_2, b_3 \ldots\ldots b_n$ respectively; such that, when their number is increased and their magnitudes diminished indefinitely, they have a constant ratio $L : 1$ each to each, so that

$$a_1 : b_1 :: L(1+\alpha_1) : 1,$$
$$a_2 : b_2 :: L(1+\alpha_2) : 1,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

where $\alpha_1, \alpha_2, \ldots\ldots$ vanish in the ultimate form of the hypothesis.

$$\therefore\ a_1 + a_2 + \ldots\ldots : b_1 + b_2 + \ldots\ldots$$

is a ratio which is intermediate between the greatest and least of these ratios, each of which is ultimately $L : 1$.

$$\therefore\ A : B :: L : 1,$$

or $A$ and $B$ are ultimately in the ultimate ratio of the parts.

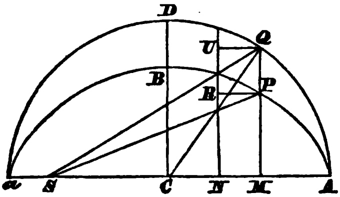
*Application of* Lemma IV *to the comparison of certain areas, and the determination of certain volumes, masses, &c.*

1. Area of an ellipse.

Let $ACa$ be the major axis of an ellipse, $BC$ the semiminor axis, $ADa$ the auxiliary circle, and let parallelograms be inscribed whose sides are common ordinates to the two curves.

Let $PMNR$, $QMNU$ be any two corresponding parallelograms.

The ratio of these parallelograms is $PM : QM$ or $BC : AC$.



Hence, by Lemma IV,

$$\text{area of ellipse} : \text{area of circle} :: BC : AC$$
$$:: \pi AC.BC : \pi AC^2,$$
$$\text{but area of circle} = \pi AC^2.$$
$$\therefore \text{area of ellipse} = \pi AC.BC.$$

**Cor. Area of a sector of an ellipse.**

If $S$ be a focus of the ellipse, and $SP$, $SQ$ be joined,

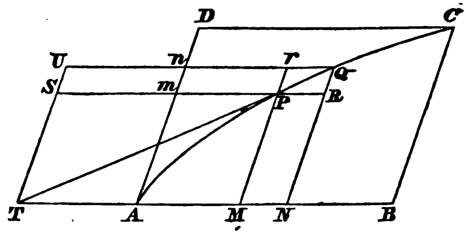
$$\Delta SPM : \Delta SQM :: BC : AC,$$
$$\text{and area } APM : \text{area } AQM :: BC : AC,$$
$$\therefore \text{area } ASP : \text{area } ASQ :: BC : AC,$$
$$\text{but area } ASQ = \Delta SCQ + \text{sector } ACQ$$
$$= \tfrac{1}{2} SC.QM + \tfrac{1}{2} AC.\text{arc } AQ;$$
$$\therefore \text{area } ASP = \tfrac{1}{2}\{SC.PM + BC.\text{arc } AQ\}.$$

2. In the following proposition it is asserted that when a chord $PQ$ is drawn to a curve from a point $P$, as $Q$ moves up to $P$, $PQ$ assumes as its limiting position that of the tangent at $P$, which is deducible from the idea of a tangent being in the direction of the curve at the point of contact.

**Area of a parabolic curve cut off by a diameter and an ordinate to the diameter.**

Let $AB$, $BC$ be the diameter and ordinate, $AD$ the tangent at $A$, $CD$ parallel to $AB$, $P$, $Q$ points near each other, $PM$, $QN$ and $Pm$, $Qn$ parallel respectively to $AD$ and $AB$.

Let $QP$ produced meet $BA$ in $T$, complete the parallograms $TMPS$, $TNQU$.




Then since $QP$ is ultimately a tangent at $P$, $AT = AM$ ultimately, and the parallelogram $PU$ is ultimately double of the parallelogram $Pn$, and the complements $PN$, $PU$ are equal;

$\therefore$ the parallelogram $PN$, $Pn$ are ultimately in the ratio $2 : 1$.

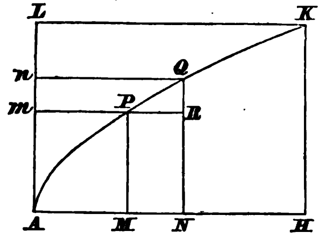
Hence, in the curvilinear areas $ABC$, $ACD$, two sets of parallelograms can be inscribed which are ultimately in the ratio $2 : 1$, each to each;

$\therefore$ area $ABC$ is ultimately double of area $ACD$,

and $\therefore$ is $\frac{2}{3}$ of the parallelogram $ABCD$.

3. Volume of paraboloid, by Lemma IV.

Let $AH$ be the axis, $AHKL$ the circumscribing parallelogram, $PN$, $Pn$ inscribed in the portions $AHK$, $AKL$.




Volume generated by $PN$
$= \pi PM^2 . MN = \pi . PM . PN.$

Volume generated by $Pn$
$= \pi QN^2 . AM - \pi PM^2 . AM$
$= \pi AM . (QN + PM) . mn$
$= \pi (QN + PM) . Pn;$

$\therefore$ vol. by $PN$ : vol. by $Pn$ :: $PM . PN : (QN + PM)\, Pn$
:: $PM . 2Pn : (QN + PM)\, Pn$ ultimately;

and $QN + PM = 2\,PM$ ultimately;

$\therefore$ vol. by $PN =$ vol. by $Pn$, ultimately;

$\therefore$ by Cor., Lemma IV,

volume generated by $AHK =$ volume generated by $AKL$,

$\therefore$ volume of paraboloid $= \frac{1}{2}$ volume of circumscribing cylinder.

## 4. Center of gravity of a paraboloid.

Since the volumes generated by $PN$ and $Pn$ are ultimately equal, the moment of the volume generated by $PN$ with respect to the tangent plane at $A$,

: moment of that generated by $Pn$,

:: distance of the center of gravity of $PN$,

: distance of center of gravity of $Pn$ ultimately;

*i.e.* :: $AM : \frac{1}{2} Pm$ ultimately,

:: $2 : 1$;

$\therefore$ the moment of volume generated by $AHK$

: that of the volume generated by $AKL$

:: $2 : 1$,

and the moment of the paraboloid

$$= \frac{2}{3} \text{ moment of the cylinder}$$

$$= \frac{2}{3} \text{ volume of cylinder} \times \frac{AH}{2};$$

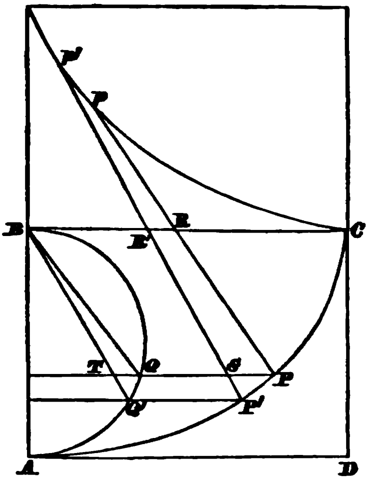
$$\therefore = \frac{2}{3} \text{ volume of paraboloid} \times AH;$$

$\therefore$ distance of center of gravity of the paraboloid from $A = \frac{2}{3} AH$.

## 5. Area of a cycloid.

Let $P$, $P'$ be two points very near each other in a cycloid, $Q$, $Q'$ corresponding points in the generating circle, $p$, $p'$ in the evolute, $R$, $R'$ the intersections of the base with normals $Pp$, $P'p'$, $T$, $S$ the intersection of $BQ'$ and $P'p'$ with $PQ$.

Then $pR = PR = BQ$, (see Appendix),




and triangle $p'RR'$ : triangle $p'PS$ :: 1 : 4, ultimately.

Also $BQT = p'RR'$, ultimately,

$BQ$, $BT$ being equal and parallel to $pR$, $p'R$;

$\therefore$ $\Delta BQT$ : $\Delta p'PS$ :: 1 : 4, ultimately,

and $\Delta BQT$ : trapezium $PRR'S$ :: 1 : 3, ultimately,

and the same being the ultimate ratio of all the inscribed triangles, and trapeziums, whose sums are ultimately the areas of the semicircle and semicycloid;

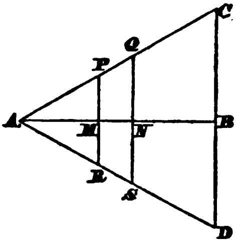
$\therefore$ by Cor., Lemma IV,

area of semicircle : area of semicycloid :: 1 : 3,

area of the cycloid = 3 times the generating circle.

6. Center of gravity and mass of a rod whose density varies as the distance from an extremity.

Let $AB$ be the rod, $MN$ a small portion of it, density at $M \propto AM$.




Construct on $AB$ as axis an isosceles triangle $CAD$, whose base is $CD$, and draw $PMR$, $QNS$ parallel to $CD$; then $PR$, $QS$, $CD$ are proportional to the densities at $M$, $N$ and $B$;

$\therefore$ the mass of $MN$ is proportional to a rectangle intermediate to rectangles $PR$, $MN$ and $QS$, $MN$ which are ultimately in a ratio of equality;

$\therefore$ the mass of $MN$ is ultimately proportional to the area of the rectangle $PR$, $MN$;

and the moment of $MN$ with respect to the line $CD$ is proportional to the moment of the same rectangle, since their distance is the same;

$\therefore$ by the Lemma, the moment of the whole rod

: the moment of the triangle with respect to $CD$

:: the mass of the rod : the area of the triangle;

$\therefore$ the distances of the centers of gravity of the rod and triangle from $CD$ are the same, and the center of gravity of the rod is at a distance $\frac{1}{3}AB$ from $B$.

Also, the mass of $MN$ being proportional to the area $PRN$, the mass of the rod is proportional to the area of the triangle $ACD$, or $\frac{1}{2}AB.CD$; $\therefore$ the mass of the rod : mass of a rod of uniform density equal to that at $B$, and of length $AB$,

$$:: \tfrac{1}{2}AB.CD : AB.CD$$

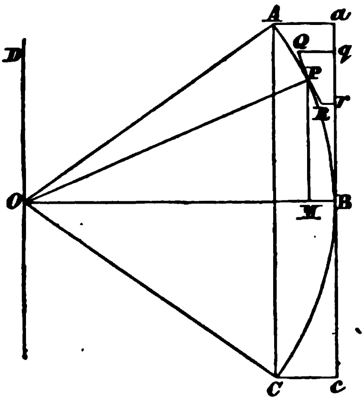
$$:: 1 : 2.$$

7. Center of gravity of a circular arc.

Let $O$ be the center of a uniform circular arc $ABC$, $OB$ the bisecting radius, $aBc$ a tangent at $B$, $OD$ parallel to $ac$, and $Aa$, $Cc$ parallel to $OB$.

Let $QR$ be the side of a regular polygon described about the arc, $P$ the point of contact $QR$, $Rr$ perpendicular to $ac$, $PM$ to $OB$, and since $OP$, $OB$ are perpendicular to $QR$, $qr$,

$$qr : QR :: OM : OP$$
$$:: OM : OB;$$




$\therefore$ since $OM$, $OB$ are the distances of the centers of gravity of $QR$ and $qr$ from $OD$, and $QR . OM = qr . OB$, the moments of $QR$ and $qr$ with respect to $OD$ are in a ratio of equality, and the same is true of every side of the circumscribing polygon;

$\therefore$ by Cor., Lemma IV, the moment of the arc, which is ultimately that of the polygon, is equal to the moment of $ac = ac . OB =$ chord $AC$. radius $OB$;

$\therefore$ the distance of the center of gravity of the arc from $O$

$$= \frac{\text{radius} \times \text{chord}}{\text{arc}}.$$

## III.

1. Find the volume of a hemisphere, by comparing the volumes generated by the quadrantal sector, and the portion of the circumscribing square which is the difference between the square and the quadrantal sector.

2. Shew that the area of the sector of an ellipse contained between the curve and two central distances, varies as the angle of the corresponding sector of the auxiliary circle.

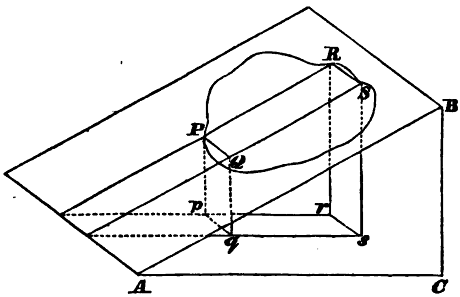
3. Find the volume of a paraboloid by comparison with the area of a triangle whose vertex and base are those of the generating parabola.

4. Find the center of gravity of the paraboloid by reference to the same triangle.

5. Find the mass of a straight rod, whose density varies as the square of the distance from the extremity, by comparison with a cone whose axis is the rod.

6. Find the volume of a paraboloid generated by the revolution of a cubical parabola, in which $PM^2 \propto AM^3$ by means of a cone on the same axis.

7. Shew that the orthogonal projection of any plane area on another plane is the given area × the cosine of the inclination of the two planes.




Prove that *pqsr* being the projection of the inscribed parallelogram *PQSR*

$$pqsr : PQSR :: \cos BAC : 1,$$

and deduce the proposition by Lemma IV.

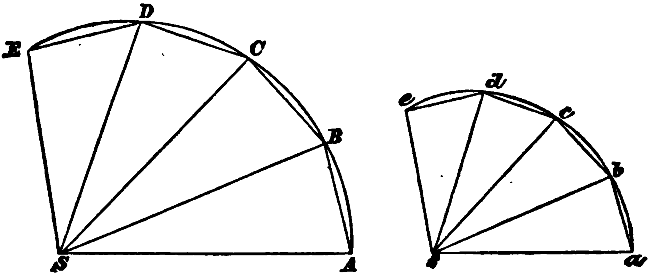
## LEMMA V.

*All the homologous sides of similar figures are proportional, whether curvilinear or rectilinear, and their areas are in the duplicate ratio of the homologous sides.*

[Similar curvilinear figures are figures whose curved boundaries are curvilinear limits of corresponding portions of similar polygons.

Let $SABCD$ ......, $sabcd$ ...... be two similar polygons of which $SA$, $AB$, $BC$, .... are homologous to $sa$, $ab$, $bc$, .... respectively.

Then (Euclid VI., Def. 1)




$$AB : SA :: ab : sa;$$

$\therefore$ alternando, $AB : ab :: SA : sa$.

Similarly, $BC : bc :: AB : ab :: SA : sa$,

$CD : cd :: BC : bc :: SA : sa$,

..............................

$\therefore$ componendo,

$$AB + BC + CD + \ldots : ab + bc + cd + \ldots :: SA : sa.$$

Now this, being true for all similar polygons, will be true in the limit, when the number of the sides $AB$, $BC$, ... and $ab$, $bc$, ... are increased, and their lengths diminished indefinitely; if, therefore, $AE$, $ae$ be curves which pass through the angular points $A$, $B$, ... and $a$, $b$, ... of the polygons, these curves are the curvilinear limits of $AB + BC +$ ... and $ab + bc +$ ... and are the

boundaries of similar curvilinear figures: and therefore
the curved line $AE$ : the curved line $ae$
$:: SA : sa :: SE : se.$

Again, (Euclid VI. 20),

polygon $SABC$ ... : polygon $sabc$ ... $:: SA^2 : sa^2$,
and this being true always, is true in the limit;
∴ (Lemma III, Cor. 2),

curvilinear area $SAE$ : curvilinear area $sac$

$$:: SA^2 : sa^2$$
$$:: AE^2 : ae^2$$
$$:: SE^2 : se^2.$$

Q. E. D.]

---

24. In order to deduce the properties of similar curves, it is premised as before mentioned under Cor. 4, Lemma III, that, if a *finite* portion of a curve be taken, and, if a polygon be inscribed in the curve, the sides of which are chords taken in order, of portions of the curve; and the number of sides of the polygon be increased indefinitely, and the magnitudes at the same time diminished indefinitely, the curve is the limit of the perimeter of the polygon. See Whewell's *Doctrine of Limits.*

It is *not* assumed that each chord is equal to the corresponding arc ultimately: this is afterwards proved for a continuous curve in Lemma VII.

*Criteria of Similarity.*

25. From the *definition* of similar curve lines, that they are curvilinear limits of homologous portions of similar polygons, or from the *test*, that

(1) "One curve line is similar to another when if *any* polygon be inscribed in one, a similar polygon can be inscribed

in the other," other tests of similarity can be deduced, which are more convenient in practice: *viz.*

(2) "If two curves be similar, and any point $S$ be taken in the plane of one curve, another point $s$ can be found in the plane of the other, such that, *any* radii $SP$, $SQ$ being drawn in the first, radii $sp$, $sq$ can be drawn in the second, having the properties that

$$\angle\, psq = \angle\, PSQ,$$

$$\text{and } sp : sq :: SP : SQ."$$

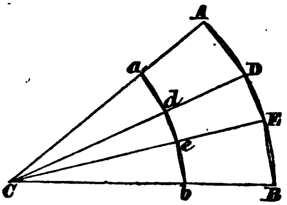
(3) "If two curves be similar, and in the plane of one curve *any* two lines $OX$, $OY$ be drawn, two other lines $ox$, $oy$ can be drawn in the plane of the other curve, inclined at the same angle, having the property that the abscissa and ordinate $OM$, $MP$ of any point $P$ in the first being taken, the abscissa and ordinate $om$, $mp$ of a corresponding point $p$ in the second will be proportional to the former, viz.,

$$om : mp :: OM : MP."$$

And the converse propositions can also be deduced, that if these proportions hold, the curves will be similar.

In order to illustrate the first test, (1).

Let the arcs of two circles, $AB$, $ab$ have the same center $C$, and the bounding radii be coincident in direction.




Let $ADEB$ be *any* polygon inscribed in $AB$ and let $CD$, $CE$ cut $ab$ in $d$, $e$; join $ad$, $de$, $eb$; these are parallel to $AD$, $DE$, $EB$, respectively,

$$\text{and } ad : de : eb :: AD : DE : EB,$$

$$\therefore\ adeb \text{ is similar to } ADEB;$$

and the arcs $ab$, $AB$ are similar.

*Deduction of criteria of Similarity.*

26. Test (2) If $ABCD \ldots$, $abcd \ldots$, be corresponding portions of similar polygons, $AB$, $BC$, ... $ab$, $bc$, ... being homologous sides, and $AS$, $BS$, ... be drawn to any point $S$, construct the triangle $sab$ equiangular with $SAB$ and join $sb$, $sc$, ... (See fig. p. 42)

$$\text{Then } sb : SB :: ab : AB$$
$$:: bc : BC,$$
$$\text{and } \angle SBC = \angle sbc;$$
$$\therefore SBC, sbc \text{ are similar triangles,}$$
$$\text{and } sc : SC :: sb : SB,$$
$$:: sa : SA;$$

and similarly for $sd$, $se$, &c.

Hence if two polygons are similar and any point be taken in one, another point can be found in the other, such that the radii drawn to corresponding angular points are proportional and contain the same angles.

If we now increase the number of sides indefinitely and diminish their magnitude, the same property holds with respect to the curvilinear limit of the polygon.

27. The converse proposition may be thus proved.

If the angles $ASB$, $BSC$, ... be equal to the angles $asb$, $bsc$, ...

$$\text{and } SA : SB : SC \ldots :: sa : sb : sc \ldots.$$

angle, and $CSP$ being common to the triangles $CPS$, $cpS$, they are similar, (Euclid VI. 7.)

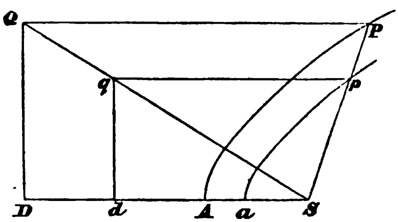
$$\therefore\ SP : Sp :: SC : Sc;$$

$\therefore$ $S$ is the center of direct similitude.

Similarly, the intersection of two common tangents which cross between the circles is a center of inverse similitude.

3. To find the condition of similarity of two conic sections, considering each as the locus of a point whose distances from a fixed point and a fixed straight line are in a constant ratio.

Let the conic sections be placed so that their directions are parallel and foci coincident, $SpP$, any line through the focus meeting them in $pP$, $SaAD$ perpendicular to the directrix $DQ$ of $AP$, $PQ$ perpendicular to $DQ$, join $SQ$ and let $pq$, parallel to $PQ$, meet it in $q$, draw $qd$ perpendicular to $SD$.




Then $\qquad SD : Sd :: Sq : SQ$

$$:: Sp : SP,$$

if the curves be similar;

and $Sp : SP$ is a constant ratio,

$\therefore$ $Sd : SD$ is a constant ratio,

and $dq$ is a fixed straight line for all positions of $p$,

and since $pq : Sp :: PQ : SP$;

$\therefore$ $pq : Sp$ is a constant ratio;

$\therefore$ $qd$ is the directrix of $ap$, and the constant ratio being the same in both, the eccentricities are the same.

4. All parabolas are similar.

For, using the last figure,

$$Sp : SP :: pq : PQ,$$

and $SP = PQ$, $\therefore$ $Sp = pq$,

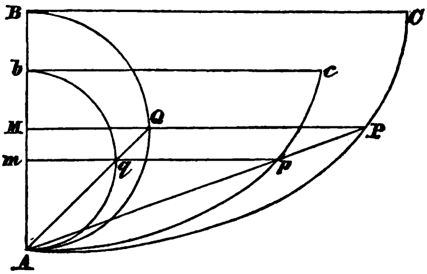
and $dq$ is the directrix of $ap$,

$$\begin{aligned} \text{and } Sp : SP &:: Sq : SQ, \\ &:: Sd : SD, \\ &:: Sa : SA; \end{aligned}$$

$\therefore$ the parabolas $ap$, $AP$ are similar.

5. Cycloids are similar.

Let two cycloids $APC$, $Apc$ be placed so that their vertices are the same, and their axes coincident in direction, and describe circles on the axes $AB$, $Ab$. Draw $AqQ$ cutting the circles in $q$, $Q$.




Then, since the segments $Aq$, $AQ$ are similar,

$$\text{arc } Aq : \text{arc } AQ :: Aq : AQ.$$

And if $mqp$, $MQP$ be ordinates to the cycloids,

arcs $Aq$, $AQ = qp$, $QP$ respectively;

$$\therefore qp : QP :: Aq : AQ,$$

and $ApP$ is a straight line.

$$\begin{aligned} \text{Also } Ap : AP &:: Aq : AQ, \\ &:: Ab : AB, \text{ a constant ratio;} \end{aligned}$$

$\therefore$ a condition of similarity is satisfied.

Obs. In this position of the cycloids the point $A$ is a center of direct similitude.

*Application of the properties of similar curves to construct curves which satisfy given conditions.*

6. **To construct a cycloid which shall have its vertex at a given point, its base parallel to a given straight line, and which shall pass through a given point.**

Let $A$ be the given vertex, $AB$ perpendicular to the given line, $P$ the given point.

In $AB$ take any point $b$, and with the generating circle, whose diameter is $Ab$, describe a cycloid $Apc$, join $AP$ intersecting this cycloid in $p$.

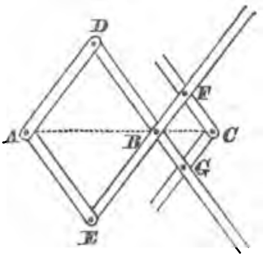
Take $AB$ a fourth proportional to $Ap$, $AP$, and $Ab$, $AB$ is the diameter of the generating circle of the required cycloid.

For, $Ap : AP :: Ab : AB$,

$\therefore$, since all cycloids are similar, $P$ is a point in the cycloid whose axis is $AB$.

7. **Instruments for copying plans on an enlarged or reduced scale are founded upon the proportion of similar figures, as the Pantagraph and the Eidograph: as also the method by dividing the plans into squares.**

The Pantagraph is an instrument for drawing a figure similar to a given figure on a smaller or larger scale, one of its forms is as in the figure;




$AD$, $EF$, $GC$ and $AE$, $DG$, $FC$

are two sets of parallel bars, joined at all the angles; at $B$ is a point, which serves to fix the instrument to the drawing board, at $A$ is a point, which is made to pass round the figure to be reduced or enlarged; at $C$ is a hole for a pencil pressed down by a weight and which traces the similar figure, altered in dimensions in the ratio of

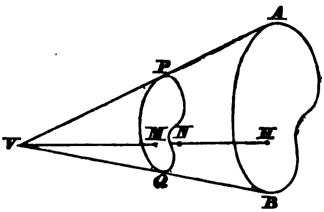
$$BC : AB, \text{ or } BF : AD.$$

The similarity of the figure traced by the pencil is a consequence of continual similarity of the triangles $ABD$, $BFC$.

By changing the positions of the pegs at $F$ and $G$ the figure described by $C$ may be made of the required dimensions.

8. Volume of a cone whose base is a plane closed figure of any form.

Let $V$ be the vertex, $AB$ the base, $VH$ perpendicular to the base from $V$: let $VH$ be divided into $n$ equal portions of which $MN$ is the $\overline{r+1}|^{\text{th}}$; and let $PQ$ be the section through $M$ parallel to $AB$.




Let $A$ be the area of $AB$.

Then if $VPA$ be *any* generating line,

$$PM : AH :: VM : VH;$$

$\therefore$ $PQ$ is similar to $AB$, and $M$, $H$ are centers of similitude,

$$\text{and area } PQ : \text{area } AB :: r^2 : n^2,$$

$$MN : VH :: 1 : n;$$

$$\therefore \text{ area } PQ \, . \, MN : A \, . \, VH :: r^2 : n^3;$$

$\therefore$ volume of cylinder whose base is $PQ$ and height $MN$

$$= \frac{r^2}{n^3} \times A \, . \, VH;$$

$$\therefore \text{ volume of cone} = A \, . \, VH \times \text{limit of } \frac{1^2 + 2^2 + \ldots\ldots + \overline{n-1}|^2}{n^3}$$

$= \frac{1}{3}$ of the cylinder whose base is $AB$ and height $VH$.

## IV.

1. Apply a criterion of similarity to shew that segments of a circle which contain the same angle are similar.

2. From the definition of an ellipse, as the locus of a point the sum of whose distances from two fixed points is constant,

shew that ellipses are similar when the eccentricities are the same.

3. Prove that the center of an ellipse is a center of inverse similitude to two opposite equal portions of the circumference of the ellipse.

4. Employ the properties of similar figures to inscribe a square in a given semicircle.

5. Shew that all the spirals of Archimedes, in which the radius vector varies as the angle, are similar.

6. Find the condition of similarity of equiangular spirals.

7. If $A$ be the vertex of a conical surface, $G$ the center of gravity of the base, $H$ that of the volume of the conical figure,

$$AH = \tfrac{3}{4} AG.$$

8. Find the center of gravity of a right cone on a circular base.

9. Deduce the position of the center of gravity of a circular sector from that of a circular arc; shew that the distance from the center is $\frac{2}{3} \cdot \frac{\text{radius} \times \text{chord}}{\text{arc}}$.

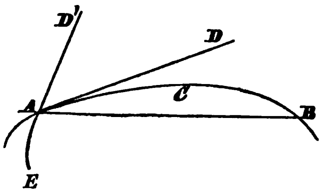
10. Shew that arcs of catenaries are similar, whose horizontal abscissæ from the lowest points are proportional to the tensions at the extremities.

11. All Lemniscates are similar.

## LEMMA VI.

*If any arc* ACB *given in position be subtended by a chord* AB, *and if at any point* A, *in the middle of continued curvature, it be touched by the straight line* AD *produced in both directions, then, if the points* A, B, *approach one another and ultimately coincide; the angle* BAD *contained by the chord and tangent, will diminish indefinitely and ultimately vanish.*

For, if that angle does not vanish, the arc $ACB$ will contain with the tangent $AD$ an angle equal to a rectilineal angle, and therefore, the curvature at the point $A$ will not be continuous, which is contrary to the hypothesis that $A$ was in the middle of continuous curvature.

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*Definitions of a tangent to a curve.*

29. (1) If a straight line meet a curve in two points $A$, $B$, and if $B$ move up to $A$, and ultimately coincide with $A$, $AB$ in its limiting position is a tangent to the curve at the point $A$.

If two portions of a curve, $EA$, and $AB$, cut one another at a finite angle in $A$, there are two tangents, $AD$, $AD'$, which are the limiting positions of straight lines $AB$ and $AE$ when $B$ and $E$ move up to $A$ along the different portions $AE$ and $AB$ of the curve respectively. And similarly if there be a multiple point in $A$ in which several branches of the curve cut one another at finite angles.

(2) The tangent is the direction of the chord of the

polygon, of which the curve is the curvilinear limit, when the number of sides are increased indefinitely.

This is founded on the same idea of a tangent as definition (1).

(3) The tangent to a curve at any point is the direction of the curve at that point.

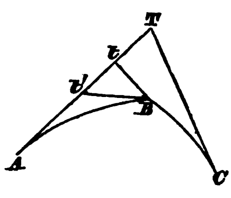
In order to apply geometrical reasoning to the tangent by employing this definition, we are obliged to explain the notion of the direction of a curve, by taking two points very near to one another, and asserting that the direction of the curve is the limiting position of the line joining these points when the distance becomes indefinitely small, which reduces this definition to the preceding.

*Observations on the Lemma.*

30. "Curvatura Continua," if we consider curves as the curvilinear limits of polygons, requires the curves to be limits of polygons whose angles continually increase as the number of the sides increase, and may be made to differ from two right angles by less than any assignable angle before the assumption of the ultimate form of the hypothesis.

If, however, as we increase the number of sides and diminish their magnitude, one of the angles remains less than two right angles by any finite difference, the curvature of the curvilinear limit is discontinuous, and the form is that of a pointed arch; in which the two portions cut one another at a finite angle.

A curve may be of continued curvature for one portion between two points, while for another its curvature changes "per saltum."




Thus, if $ABC$ be a curve forming at $B$ a pointed arch, it may be of continued curvature from $B$ to $A$ and from $C$ to $B$, though not from $C$ to $A$.

In this case the tangents in passing from $C$ to $A$ assume all positions interme-

diate to $CT$, $Bt$, and $Bt'$, $TA$, but at $B$ they pass from $Bt$ to $Bt'$ without assuming the intermediate positions.

31. "In medio curvaturæ continuæ," implies that the point $A$ in the enunciation of the Lemma is not such a point as $B$ in the last figure, but in passing from a point on one side of $A$ to another on the other side, the tangents pass through all the intermediate positions.

The curvature is supposed to be in the same direction in the figure of the Lemma, which in all curves of continued curvature is possible, if $B$ be taken sufficiently near to $A$ at the commencement of the change in the construction.

If the point $A$ be not "in medio curvaturæ continuæ," two tangents $AD$, $AD'$ may be drawn at $A$ to the two parts of the curve, and the curve $BCA$ makes a finite angle with one of the tangents $AD'$.

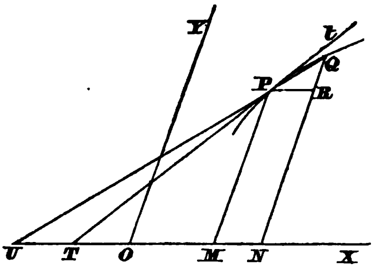
But, even in this case, the angle between the chord and that tangent which belongs to the portion of the curve considered, continually diminishes and ultimately vanishes.

*Definition of the subtangent.*

32. The part of the line of abscissæ intercepted between the tangent at any point and the foot of the ordinate of that point is called the subtangent.

*Methods of finding tangents to curves.*

33. Let $OM$, $MP$ be the abscissa and ordinate of a point $P$ in a curve, and let $Q$ be a point near $P$, $ON$, $NQ$ its abscissa and ordinate.




Let $QPU$ meet $OY$ the line of abscissæ in $U$; then, if $PR$ parallel to $OM$ meet $QN$ in $R$;

$$PM : MU :: QR : PR$$
$$:: QN - PM : ON - OM.$$

Now as $Q$ approaches to $P$, the limiting position of $QPU$ is that of the tangent at $P$, (Lemma VI.) viz. $tPT$,

and $PM : MT$ is the limiting ratio of

$$QN - PM : ON - OM.$$

This ratio determines the position of the tangent at $P$, and, if the ordinates be perpendicular to the abscissæ, is the trigonometrical tangent of the angle made by the tangent with the line of abscissæ.

$MT$ is the subtangent.

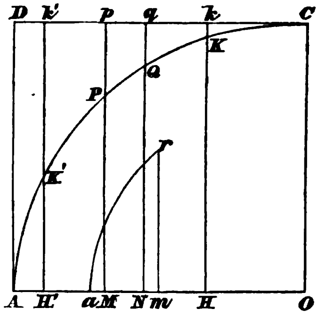
*Illustrations.*

1. In the common parabola, in which

$$PM^2 : QN^2 :: OM : ON;$$
$$\therefore\ QN^2 - PM^2 : PM^2 :: ON - OM : OM,$$
$$\text{and } QN - PM : PM :: ON - OM : MT,$$
$$QN + PM : PM :: 2 : 1, \text{ ultimately},$$
$$\therefore\ QN^2 - PM^2 : PM^2 :: 2(ON - OM) : MT,$$
$$\therefore\ MT = 2\,OM.$$

## 2. Surface of a segment of a sphere.

Let $AKH$ be the portion of a circle which generates by revolution round $AH$ the spherical segment, $O$ the center of the circle, $PQ$ the chord of a small arc, $PM$, $QN$ perpendicular to $AH$.




Let $AOCD$ be the rectangle circumscribing the quadrant, and generating the circumscribing cylinder.

Produce $MP$, $NQ$, $HK$ to $CD$ in $p$, $q$, $k$. Since $PQ$ is in its limiting position a tangent at $P$, $PQ$ is ultimately perpendicular to the radius $OP$ and $pq$ to $MP$;

$$\therefore PQ : pq :: OP : PM, \text{ ultimately},$$

and the surface generated by $PQ$ is ultimately $2\pi PM.PQ$ (Art. 17),

$$= 2\pi . OP . pq = \text{surface generated by } pq.$$

The same is true for each side of the inscribed polygon, when the number is indefinitely increased.

$\therefore$ the surface generated by $AK$ or the surface of the spherical segment is equal to the surface of the circumscribed cylinder cut off by the plane of the base of the segment.

Cor. 1. Hence also, the surface of any belt of a sphere cut off by two parallel planes is equal to the corresponding belt of the cylindrical surface.

Cor. 2. The moment of the belt generated by $PQ$ with respect to the plane of contact of the sphere and cylinder, generated by $OD$, is evidently ultimately equal to that of the belt generated by $pq$;

$\therefore$ the moment of any belt generated by $K'K$ is equal to that of the corresponding belt by $k'k$;

$\therefore$ the centers of gravity of the two belts are coincident, viz. in the bisection of $HH'$, or, the distance of the center of gravity of a spherical belt, contained between parallel planes is half-way between the two planes.

COR. 3. A spherical sector is generated by the revolution of a sector $AOP$.

The volume of the spherical sector is equal to the limit of the sum of a series of pyramids whose vertices are in $O$, and the sum of whose bases is ultimately the area of the surface of the segment,

and the volume of each pyramid is $\frac{1}{3}$ base × altitude;

∴ volume of the spherical sector is, by Art. 18,

$$\frac{1}{3} \text{ area of the surface of the spherical segment} \times \text{radius}$$
$$= \tfrac{1}{3} . 2\pi . AD . Dp . AO$$
$$= \frac{2\pi}{3} . AM . AO^2$$
$$= \frac{2\pi AO^3}{3} \text{ vers } POA.$$

COR. 4. If we suppose each of the pyramids on equal bases, they may be supposed collected in their centers of gravity, whose distances are $\frac{3}{4} AO$ from $O$ ultimately, and they form a mass which may be distributed uniformly over the surface of a spherical segment whose radius is $\frac{3}{4} AO$, viz. that generated by $ar$, whose center of gravity is in the bisection of $am$;

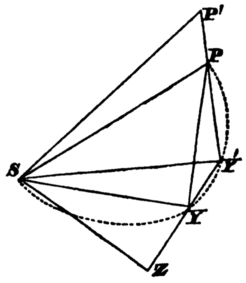
∴ the distance of the center of gravity of the spherical sector from $O$

$$= \tfrac{1}{2}(Oa + Om)$$
$$= \tfrac{1}{2} Oa (1 + \cos rOa)$$
$$= \tfrac{3}{4} OA . \cos^2 \tfrac{1}{2} POA.$$

If the angle $POA$ become a right angle, the distance of the corresponding sector which becomes the hemisphere is $\frac{3}{8} OA$.

3. If $SY$ be the perpendicular on the tangent $PY$ at $P$ in a curve, $Y$ will trace out a curve, and if $YZ$ be a tangent to the locus of $Y$, $SZ$ perpendicular to it,

$$SY^2 = SP . SZ.$$




Let $P'$ be a point near $P$,

$SY'$ perpendicular on $P'P$,

$SZ$ perpendicular on $Y'Y$.

Since angles $SYP$, $SY'P$ are right angles, a semicircle on $SP$ passes through $Y$, $Y'$;

∴ the angles $SY'Y$, $SPY$, in the same segment, are equal,

and the right angles $SZY'$, $SYP$ are equal;

$\therefore$ the triangles $SPY$, $SY'Z$ are similar,

and $SZ : SY' :: SY : SP$,

but, ultimately, as $P'$ moves up to $P$, $P'PY'$ becomes the tangent at $P$,

and $Y'YZ$ that at $Y$ to its locus,

and $SY' = SY$;

$\therefore SZ . SP = SY^2$.

Q. E. D.

## V.

1. In the curve in which the abscissa varies as the cube of the ordinate, shew that the subtangent is three times the abscissa.

2. In the logarithmic curve in which the abscissa varies as the logarithm of the ordinate shew that the subtangent is constant.

3. If $PY$ a tangent to an ellipse at $P$ meet the auxiliary circle at $Y$, and $ST$ be perpendicular to the tangent at $Y$, $ST$ varies inversely as $HP$.

4. $AB$ is the diameter of a semicircle $AQB$, in which $AM$ is taken equal to $BN$, $QN$ is an ordinate, $AQ$ meets the ordinate corresponding to $AM$ in $P$, the locus of $P$ is the cissoid, shew that the subtangent at $P : AM :: 2AN : 2AN + AB$.

5. In the Lemniscate if $SY$ be perpendicular to the tangent at $Q$, $SA$ being the greatest value of $SQ$ shew that

$$SQ^3 = SY . SA^2.$$

Apply illustration 3, and the properties proved in the Appendix.

6. In the catenary, if the line of abscissæ be at a vertical distance from the lowest point equal to the length of the string

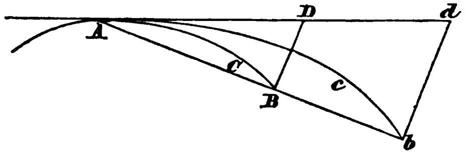
whose weight is the tension at the lowest point $A$, shew that the subtangent at any point $P$ : horizontal distance of $P$ from $A$

:: the tension at $P$ : the tension at $A$.

## LEMMA VII.

*If any arc, given in position, be subtended by the chord* AB, *and at the point* A, *in the middle of continued curvature, a tangent* AD *be drawn, and the subtense* BD, *then, when* B *approaches to* A *and ultimately coincides with it, the ultimate ratio of the arc, the chord, and the tangent to one another is a ratio of equality.*

For whilst the point $B$ approaches to the point $A$, let $AB$, $AD$ be supposed always to be produced to points $b$ and $d$ at a finite distance, and $bd$ be drawn parallel to the subtense $BD$, and let the arc $Acb$ be always similar to the arc $ACB$, and having, therefore, $ADd$ for its tangent at $A$.

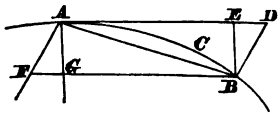



But, when the points $B$, $A$ coincide, the angle $bAd$ by the preceding Lemma, will vanish, and therefore, the straight lines $Ab$, $Ad$ which are always finite and the arc $Acb$ which lies between them [and is of continuous curvature,] will coincide ultimately, and therefore will be equal.

Hence also, the straight lines $AB$, $AD$ and the intermediate arc $ACB$ which are always proportional to them, will vanish, and have an ultimate ratio of equality to one another.

COR. 1. Hence, if through $B$, $BF$ be drawn parallel to the tangent, always cutting any straight line $AF$ passing through $A$ in $F$, this $BF$ will have ultimately to the vanishing arc $ACB$ a ratio of equality, since, if the parallelogram $AFBD$ be completed, it has always a ratio of equality to $AD$.

COR. 2. And if, through $B$ and $A$ be drawn many straight lines $BE$, $BD$, $AF$, $AG$ cutting the tangent $AD$ and $BF$, parallel to it; the ultimate ratio of all the abscissæ $AD$, $AE$, $BF$, $BG$ and of the chord and arc $AB$ to one another will be a ratio of equality.




COR. 3. And, therefore, all these lines in every argument concerning ultimate ratios may be used indifferently one for the other.

34. The subtense of the angle of contact of an arc is a straight line drawn from one extremity of the arc to meet, at a finite angle, the tangent to the arc at the other extremity.

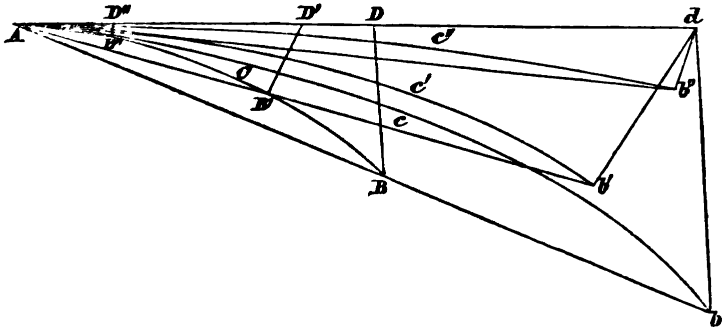
This subtense is the secant defining the limited line called, in the Lemma, "the tangent."

The chord is called by Newton "the subtense of the arc," see Lemma XI.

*Observations on the Lemma.*

35. In the construction for this Lemma, $BD$ must be a subtense, *i. e.* inclined throughout the change of position at a finite angle to the tangent or chord, for, otherwise, the angles $BAD$ and $ABD$ being both small, the ultimate ratio of the chord to the tangent might be any finite ratio instead of being one of equality.

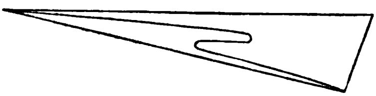
This is the only limitation of the motion of $BD$; the following figure represents changes which may take place in the approach towards the ultimate state of the hypothesis;




*b*, *d* are the distant points, *i. e.* at a finite distance from *A*; *BD*, *B'D'*, *B''D''* positions of the subtense, when *B* approaches towards *A*, *db*, *db'*, *db''*, parallel to these.

*Ac'b'*, *Ac''b''*, the *changed* forms of *Acb* so as to be always similar to the portion of *ACB* cut off by the chord.

36. It should be remarked that the curve *Acb* is not intermediate in *magnitude* to the two lines *Ab*, *Ad* but only in *position*, for example, *Ab* may be equal to *Ad*, if *BD* make equal angles with the two lines, and the curve line is greater than either *Ab* or *Ad*; but it becomes in all cases less bent, until it is ultimately rectilinear, and the three *Acb*, *Ab*, *Ad* ultimately equal; the only alternative being that the curve becomes doubled up as in the figure, which is precluded by the supposition that the curvature is continued, in the same direction, near *A*, throughout the passage from *B* to *A*.



37. The subtense ultimately vanishes compared with the arc.

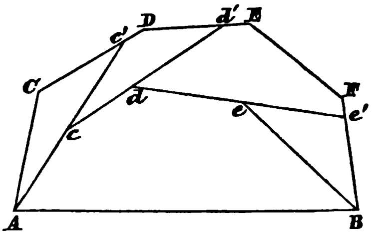
For $BD : ACB :: bd : Acb$,

and since *bd* vanishes, and *Acb* remains finite, in the limit, the

ratio $BD : ACB$ vanishes ultimately. In curves of finite curvature it will be afterwards seen that $BD \propto ACB^2$.

38. **If two curves of continued curvature which do not intersect have a common chord, the length of the exterior curve is greater than that of the interior.**

Let $AcdeB$, $ACDEFB$ any two polygons, having a common side $AB$, be such, that the first lies entirely within the second, and that neither has internal angles, the perimeter of the first is less than that of the second.




For, produce $Ac$, $cd$, $de$ to meet the perimeter of the exterior in $c'$, $d'$, $e'$.

Then $AC + Cc' > Ac'$;

$$\therefore\ ACDEFB > Ac'DEFB.$$

Similarly $$Ac'DEFB > Acd'EFB,$$

and so on;

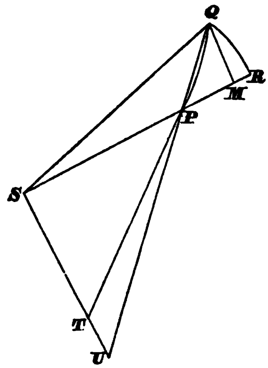
$$\therefore,\ \text{a fortiori},\ ACDEFB > AcdeB.$$

And, since the same is true in the limit when the number of sides is increased indefinitely, therefore the curvilinear limits of the polygons have the same property, and the proposition is proved.

*The polar subtangent and the inclination of the tangent to the radius vector, at any point of a spiral.*

39. Let $S$ be the pole, $PT$ the tangent at a point $P$ in the curve, $ST$ perpendicular to $SP$ meeting $PT$ in $T$ is the polar subtangent, for the point $P$.

Let $Q$ be a point near $P$, $QM$ perpendicular to $PS$, produced if necessary, $QR$ the circular arc, center $S$, meeting $SP$ in $R$.




Let $QP$ meet $ST$ in $U$, then

$$SU : SP :: QM : PM,$$

and $MR : QM :: QM : SM + SR$, and when $Q$ approaches indefinitely near to $P$, $QM$ vanishes compared with $SM + SR$;

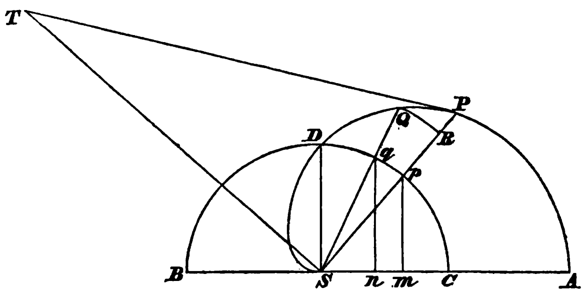
$\therefore$ $MR$ vanishes compared with $QM$ or $PM$;

$$\therefore\ SU : SP :: QM : PR \text{ ultimately};$$

$\therefore$ $ST : SP$ is the limiting ratio of $QR : PR$; or $QR : SQ \sim SP$.

Hence $ST$, and also the trigonometrical tangent of the angle $SPT$ between the tangent and the radius vector are known.

*Inclination of the tangent to the radius vector in the Cardioid.*




If $Bqp$ be a circle whose center is $S$ and diameter $BC$, $pm$ an ordinate at $p$, produce $Sp$ to $P$, making $SP = Bm$, $P$ traces out the cardioid $APS$, making the same construction as before,

$$ST : SP :: QR : SQ \sim SP \text{ ultimately};$$

and if $SQ$ meet the circle in $q$, draw the ordinate $qn$,

then, $SP - SQ = mn$;

and $QR : pq :: SQ : Sq$

$:: SP : Sp$ ultimately;

also $pq : mn :: Sp : pm$ ultimately;

$\therefore\ QR : SP - SQ :: SP : pm$ ultimately;

$\therefore\ ST : SP :: Bm : pm$;

$\therefore\ \angle\, SPT = \angle\, mpB =$ complement of $\frac{1}{2} \angle\, PSA$;

whence the cardioid cuts $SA$ at right angles at $A$, touches $SB$ at $S$, and cuts the circle at an angle equal to half a right angle.



## VI.

1. $RQq$ is a common subtense to two curves $PQ$, $Pq$ which have a common tangent $PR$ at $P$. When $RQq$ approaches to $P$, $RQ$ and $Rq$ ultimately vanish, is the ratio $RQ : Rq$ ultimately a ratio of equality?

2. Prove that the circular measure of an angle lies between the trigonometrical sine and tangent of the angle.

3. In the hyperbolic spiral, in which the radius vector varies inversely as the spiral angle, prove that the subtangent is constant.

4. In the spiral of Archimedes, in which the radius vector varies directly as the angle, shew that the subtangent is equal to the arc of the circle, whose radius is the radius vector which is subtended by the spiral angle.

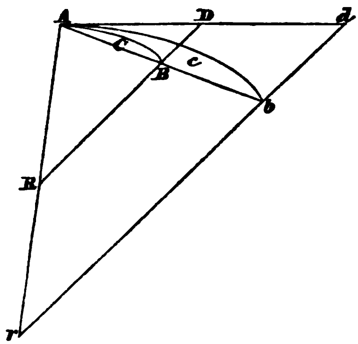
5. Shew that the extremity of the polar subtangent from the focus of a conic section is always in a fixed straight line.

6. In any curve, if $Q$ be the intersection of perpendiculars to two consecutive radii vectors, through their extremities, and $SY$ be the perpendicular from the pole $S$ on the tangent at $P$, prove that ultimately $SP^2 = SY \cdot SQ$.

## LEMMA VIII.

*If two straight lines* AR, BR, *make with the arc* ACB, *the chord* AB, *and the tangent* AD, *the three triangles* RACB, RAB, *and* RAD, *and the points* A, B *approach one another; then the ultimate form of the vanishing triangles is one of similitude, and the ultimate ratio one of equality.*

For, whilst the point $B$ is approaching the point $A$, let $AB$, $AD$, $AR$ be always produced to points $b$, $d$, $r$ at a finite distance, and $rbd$ be always drawn parallel to $RD$, and let the arc $Acb$ be always similar to the arc $ACB$, and therefore have $Dd$ for the tangent at $A$. Then, when the points $B, A$ coincide the angle $bAd$ will vanish, and therefore the three triangles $rAb$, $rAcb$, $rAd$, will coincide, and are therefore in that case similar and equal. Hence also, $RAB$, $RACB$, $RAD$, which are always similar and proportional to these, will be ultimately similar and equal to one another.




Cor. And hence, in every argument concerning ultimate ratios, these triangles can be used indifferently for one another.

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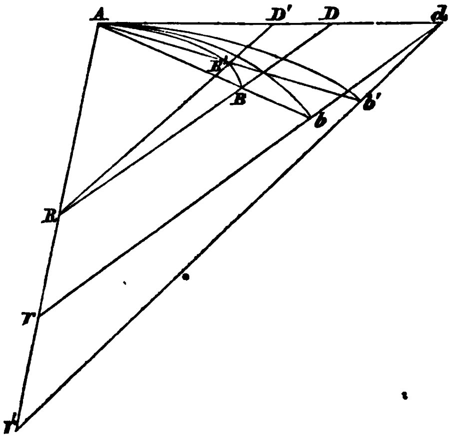
*Observations on the Lemma.*

40. If $RB$ throughout the change in the hypothesis make a finite angle with $RA$, the three triangles $rab$, $racb$, $rad$ remain always finite and are ultimately identical and equal. But,

if the angle $ARB$ is ultimately not finite, for example, if $RB$ revolve round a fixed point $R$, the $rab,\ldots$ triangle become infinite, since $r$ moves to $r'$ and so on to an infinite distance, and




there is as great an objection to dealing with these infinite triangles, as to reasoning immediately upon the relation of the triangles $RAB$, $RAD$ in the former case.

In this case we can at once deduce the equality of the triangles without producing $AD$ to a point $d$ at a finite distance. For, the ratio of the difference of $RAD$ and $RAB$ to $RAB$ is $BD : RD$, which vanishes ultimately, since $RD$ is finite in this case; $\therefore$ $RAB$ and $RAD$ and the curvilinear triangle, which is intermediate in magnitude, are ultimately in a ratio of equality.

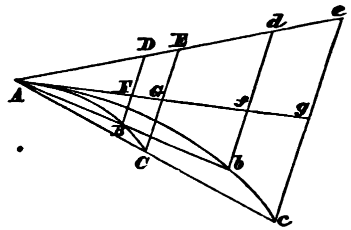
## LEMMA IX.

*If a straight line* AE *and curve* ABC, *given in position, cut one another in a finite angle* A, *and ordinates* BD, CE *be*

***drawn, inclined at another finite angle to that straight line, and meeting the curve in* B, C; *then, if the points* B, C *move up together to the point* A, *the areas of the curvilinear triangles* ABD, ACE, *will be ultimately to one another in the duplicate ratio of the sides.***

**For, as the points $B$, $C$ are approaching the point $A$, let $AD$, $AE$, be always produced to the points $d$, $e$ at a finite distance, such that**

$$Ad : Ae :: AD : AE,$$




**and let the ordinates $db$, $ec$ be drawn parallel to $DB$, $EC$ meeting the chords $AB$, $AC$ produced in $b$, $c$, [and $Ab : AB :: Ad : AD$**

$$:: Ae : AE :: Ac : AC;$$

$$\therefore\ Ab : Ac :: AB : AC,]$$

**hence a curve $Abc$ can be supposed to be drawn always similar to $ABC$, while $B$ and $C$ move up to $A$.**

**Let the straight line $Ag$ be drawn touching both curves at $A$, and cutting the ordinates $DB$, $EC$, $db$, $ec$ in $F$, $G$, $f$, $g$.**

**[Now areas $ABD$, $abd$, by Lemma V, are always in the duplicate ratio of $AD$, $ad$, and areas $ACE$, $ace$ in the duplicate ratio of $AE$, and $Ae$,**

$$\text{and } AD : Ad :: AE : Ae;$$

$$\therefore\ ABD : Abd :: ACE : Ace;]$$

**If, then, the points $B$ and $C$ move up to $A$ and ultimately coincide with it, the angle $cAg$ will ultimately vanish, and the curvilinear areas $Abd$, $Ace$ will coincide with the rectlinear triangles $Afd$, $Age$, and therefore will be ultimately in the duplicate ratio $Ad$, $Ae$.**

But $ABD$, $ACE$ are proportional to $Abd$, $Ace$, always,
also $AD$, $AE$ are proportional to $Ad$, $Ae$;
$\therefore$ also areas $ABD$, $ACE$ are ultimately in the duplicate ratio of $AD$, $AE$.

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*Observations on the Lemma.*

41. By a finite angle is to be understood an angle less than two right angles, and neither indefinitely small nor indefinitely near to two right angles.

The angles between $AD$ and the curve and between $AD$ and $BD$ are different finite angles, because otherwise $BD$ would not meet the curve.

42. It is not necessary that $d$ and $e$ be fixed, but only that they remain at a finite distance from $A$, and that the proportion be retained.

The student, by reference to Arts. 35, 40, will be able to exhibit the change in the figure which will correspond to a change of the position of $B$ and $C$ in the progress towards the ultimate position.

43. When the angle $CAG$ vanishes, the curvilinear areas $Abd$, $Ace$ coincide with the rectilinear triangles $Afd$, $Age$, and so are in the duplicate ratio of $Ad : Ae$. But if the angle $DAF$ be not finite those triangles will not themselves be finite, and the object aimed at by producing to a finite distance will not be attained.

The fact is, that the triangle $Adb$ is made up of the triangle $Adf$ and the curvilinear triangle $Afb$, of which the latter is indefinitely small ultimately, and the former finite; therefore $Afb$ vanishes compared with $Adf$, but if $Adf$ be indefinitely small, the ratio $\Delta AFB : \Delta AGC$ must be found by another process, and by

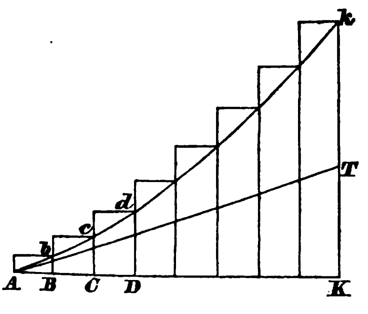
referring to Lemma XI, it will be found that the ratio is that of cubes of the arcs ultimately.

If the angle $DAF$ be greater than a right angle, the figure assumes a form in which $AD$ lies below $ABC$, hence $DB$, $EC$,... must be produced to meet the tangent, and the argument proceeds the same as before.

## LEMMA X.

*The spaces which a body describes [from rest] under the action of any finite force, whether that force be constant or else continually increase or continually diminish, are in the very beginning of the motion in the duplicate ratio of the times.*

[Let the times be represented by lines measured from $A$, along $AK$ and the velocities generated at the end of those times, by lines drawn perpendicular to $AK$. Suppose the time represented by $AK$ to be divided into a number of equal intervals, represented by $AB$, $BC$, $CD$,... and $Bb$, $Cc$, $Dd$. ... $Kk$ represent the velocities generated in the times $AB$, $AC$,... $AK$ respectively, and let $Abcd$... be the curve line, which always passes through the extremities of these ordinates. Complete the parallelograms $Ab$, $Bc$, $Cd$,...




In the interval of time denoted by $CD$, the velocity continually changes, from that represented by $Cc$, to that represented by $Dd$, and therefore, the space described

in that time is intermediate between the spaces represented by the parallelograms $Dc$ and $Cd$; therefore the spaces described in the times $AD$, $AK$ are represented by areas which are intermediate between the sums of the parallelograms inscribed in and circumscribed about the curvilinear areas $ADd$ and $AKk$ respectively.

Therefore, by Lemma II, the number of intervals being increased, and their magnitudes diminished indefinitely, the spaces in the times $AD$, $AK$ are proportional to the curvilinear areas $ADd$, $AKk$.

Now, the force being finite, the ratio of the velocity to the time is finite, $\therefore$ $Kk : AK$ is a finite ratio, however small the time be taken; $\therefore$ if $AT$ be the tangent to the curve line at $A$, meeting $Kk$ in $T$, $KT : AK$ is a finite ratio; $\therefore$ the angle $TAK$ is finite, or $AK$ meets the curve at a finite angle.

Hence, by Lemma IX, if $AD$, $AK$ be indefinitely diminished,

$$\text{area } ADd : \text{area } AKk :: AD^2 : AK^2;$$

$\therefore$ in the beginning of the motion, the spaces described are proportional to the squares of the times of describing them. Q.E.D.]

COR. 1. And hence it is easily deduced, that the errors of bodies, describing similar parts of similar figures in proportional times, which are generated by any equal forces acting similarly upon the bodies, and which are measured by the distances of the bodies from those points of the similar figures, to which the same bodies would have arrived in the same proportional times without the action of the disturbing forces, are approximately as the squares of the times in which they are generated.

Cor. 2. But the errors which are generated by proportional forces, acting similarly at similar portions of similar figures, are approximately as the forces and the square of the times conjointly.

Cor. 3. The same is to be understood of the spaces which bodies describe under the action of different forces. These are, in the beginning of the motion conjointly, as the forces and the squares of the times.

Cor. 4. Consequently, in the beginning of the motion the forces are as the spaces described directly, and the squares of the times inversely.

Cor. 5. And the squares of the times are as the spaces described directly and the forces inversely.

---

The proof given in the original Latin is as follows:

Exponantur tempora per lineas $AD$, $AE$, et velocitates genitæ per ordinatas $DB$, $EC$; et spatia, his velocitatibus descripta, erunt ut areæ $ABD$, $ACE$ his ordinatis descriptæ, hoc est, ipso motus initio (per Lemma IX.) in duplicata ratione temporum $AD$, $AE$. Q. E. D.

---

44. This proof has been amplified, in order to exhibit in what manner the descriptions of areas by the flux of the ordinates, corresponds to that of spaces by the velocities represented by the ordinates; also, to shew the propriety of the application of the ninth Lemma, by reference to the definition of finite force, which may be stated as follows:

"A force is finite when the ratio of the velocity generated in any time to the time in which it is generated, is finite, however small the time be taken."

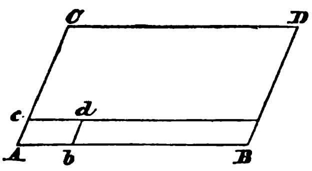
*Observations on the Lemma.*

45. In the proof of this Lemma, time is represented by the length of a straight line, and a distance traversed by a body is represented by an area.

If the length of a straight line be always proportional to the period of time elapsed, the straight line is a proper representation of the time. Thus $n$ inches has the same ratio to one inch that $n$ seconds has to one second; and on this scale the length $n$ inches is a proper representation of $n$ seconds.

If an area is always in the same ratio to the unit of area that the length of a straight line is to the unit of length, the area is a proper representation of the length of the straight line.

Thus, if $Ab$ be one foot, $AB$, $n$ feet, $Ac$ an inch, and $AC$, $t$ inches: complete the parallelograms $ABDC$, $abdc$, and $Bc$, $ABCD$ contains $nt$ such areas as $abdc$.




If now a particle move with a uniform velocity of $n$ feet a second, $AC$ represents $t$ seconds on the scale of one inch to a second; the parallelogram $Bc$ represents the space travelled over in the first second, since it contains $n$ times the parallelogram $Abdc$, and $ABDC$ represents the space travelled over in $t$ seconds.

There will be no difficulty in the representation of a period of time by a line, or of a distance by an area, if the student bears in mind that periods of time and lengths of lines, although existing absolutely, are only estimated by their ratios to certain standard periods, and standard lengths, and they are therefore determined whenever these ratios are given, which may be given either directly in numbers or by the comparison of any magnitudes whatever *of the same kind.*

46. COR. 1, 2. If bodies describe orbits under the action of certain forces, and small forces extraneous to those under the

action of which the orbits are described, be supposed to act upon the bodies, the orbits are disturbed slightly, and the errors spoken of are the linear disturbances of the bodies, at any time, from the positions which they would have occupied at that time, if the extraneous forces had not acted.

Thus, in calculating the motion of the Moon considered as moving under the attraction of the Sun and Earth, it is convenient to estimate the motion which she would have subject to the attraction of the Earth alone, and then to calculate what will be the disturbing effect of the Sun upon this orbit.

47. If $AB$ be a portion of an orbit described by a body in any time, $AC$ the portion of the orbit described when a disturbing force is introduced, $BC$ is "quam proxime" the space which would have been described in the same time from rest by the action of the disturbing force alone. When the time is taken small but not indefinitely small, the expression, in the statement of the corollaries, "approximately," is necessary for two reasons; for, in the first place, the position of the body in space is not the same at the end of any interval in the lapse of the time as if the body had moved from rest under the action of the disturbing force alone, and therefore the magnitude of the force is not the same generally either in direction or magnitude; and, in the second place, since the force is not generally uniform, the variation according to the duplicate ratio of the times is not exact, except in the limit.

But when the times are taken very small the variation of direction and magnitude of the force may be neglected as an approximation to the true state of the case.

48. Application of the method of Lemma X to determine the space described in a finite time from rest by a particle under the action of a *constant* force.

In this case, since the acceleration is constant, the velocity varies as the time.

Hence, the curve $Ak$ is a straight line, because the ordinates vary as the abscissæ.

Therefore, the space which is described in the time represented by $AK$ is represented by the area of the triangle $AKk$, and the space which would be described uniformly in the same time with velocity acquired at the end of that time is represented by the rectangle whose diagonal is $Ak$, or twice the area of the triangle $AKk$;

$$\therefore \text{ space in time } t = \frac{Vt}{2} = \frac{ft^2}{2},$$

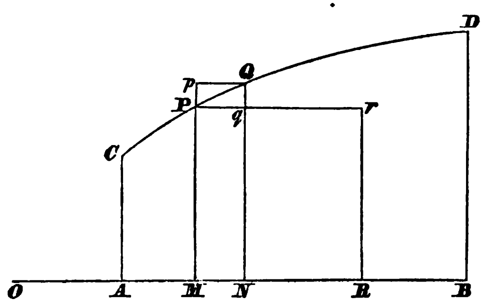
where $V$ = velocity at the time $t$,

and $f$ = acceleration in an unit of time.

49. *General geometrical representation of the space described by a body in a finite time when it moves with a variable velocity.*

Prop. If a curve be found, such that the ordinate at each point represents the velocity corresponding to a time represented by the abscissa, then the space described by the body will be represented by the area bounded by the curve, the line of abscissæ, and the ordinates corresponding to the commencement and end of the time of motion.

Let $OA$, $OB$ represent the times at the commencement and end of the interval during which the motion of the body is to be



examined. Let $OM$ be any other time, and let $AC$, $MP$, $BD$ represent the velocity at the end of the times represented by $OA$, $OM$, $OB$; $CPD$ the curve which passes through the extremities of all such ordinates as $MP$.

Let $AB$ be divided into any number of small portions, such as $MN$; $NQ$ the ordinate corresponding to $ON$. Complete the parallelograms $PMNq$, $QNMp$, and suppose corresponding parallelograms to be constructed on all the bases corresponding to $MN$.

The body during the time represented by $MN$ moves with a velocity, which if $MN$ is taken small enough is intermediate in magnitude to the velocities represented by $PM$ and $QN$, and the space described during that time is intermediate in magnitude to the spaces which would have been described with uniform velocity equal to those represented by $PM$ and $QN$, or to the spaces represented by the areas $PN$, $QM$.

Hence the whole space described in the interval of time represented by $AB$ is greater than that represented by the inscribed series, and less than that by the circumscribed series of parallelograms, which, by the Lemma II, are ultimately equal to the area $ACDB$, when the number of portions into which $AB$ is divided is indefinitely increased, and their magnitudes diminished; therefore the proposition is proved.

50. Cor. 1. The velocity is the limit of the ratio of the space to the time when the time is indefinitely diminished.

The velocity $V$ at the time $OM$ is represented by $MP$, therefore, if $T$ be the time represented by $MN$, $VT$ : space described in time $T$ :: $MP \,.\, MN$ : area $PM \,.\, NQ$, but $MP \,.\, MN$ = area $PMNq$ = area $PMNQ$, ultimately; therefore $VT$ = space described in time $T$, ultimately. Whence the truth of the proposition.

51. Cor. 2. The velocity is measured by the space which

would be described in an unit of time if the velocity remained uniform during this time.

Let *MR* represent the unit of time. Complete the parallelogram *PMRr*. Then *PMRr* represents the space described in an unit of time, with the velocity at time *OM* continued uniform, and since *MR* is constant, therefore *PMRr* varies as *PM*; therefore the velocity is properly represented by *PMRr*, and the proposition is proved.

52. *Geometrical representation of the velocity generated by a finite and variable force in a given time.*

Prop. If a curve be found such that the ordinate at each point represents the accelerating effect of the force corresponding to a time represented by the abscissa, then the velocity generated in a body in a given time, moving in the direction of the force, will be represented by the area bounded by the curve, the line of abscissæ, and the ordinates corresponding to the commencement and end of the time considered.

The proof proceeds in a manner similar to that given in (49). The student can supply it, employing the same figure, in which the ordinates now represent the accelerating effect of the force at the times represented by the corresponding abscissæ, and observing that the motion of the body is accelerated during the time represented by *MN* by a force whose accelerating effect is intermediate in magnitude to those represented by *PM* and *QN*, if *MN* is taken small enough, and the velocity generated is intermediate to those which would have been generated by uniform forces equal to those whose accelerating effects are represented by *PM*, *QN*, or to the velocities represented by the areas *PN*, *QM*.

53. And as before, the force at any time is measured by the limit of the ratio of the velocity generated to the time in which it is generated.

54. Also, the force at any time is measured by the velocity which would be generated in an unit of time, if the force continued uniform during that time, and equal to the force at the given time.

55. *Geometrical representation of the velocity generated by a force, which acts upon a body moving in the direction of the force's action, when the force is described as depending in any manner upon the* distance *from any fixed point in that direction.*

In the last figure, let $OAB$ be the line of motion of the body, $O$ a fixed point in this line, and when it arrives at a point $M$, let $MP$ be taken to represent the accelerating effect of the force acting upon it; let $CPD$ be a curve whose ordinates represent the accelerating effect of the force for the different positions of the body at the foot of the ordinates.

Let $AB$ be the space traversed by the body, and let it be divided into any number of small portions, of which suppose $MN$ one, and let $QN$ be the ordinate at $N$, $PMNq$, $QNMp$ complete parallelograms.

If during the time occupied in describing $MN$ the force remained constant, the difference of the squares of the velocities at $M$ and $N$ would be represented by $2MN.PM$ or $2MN.QN$ or by twice the parallelograms $2PN$ or $2QM$ according as the uniform force was that represented by $PM$ or $QN$.

Hence the difference of the squares of the velocities at $M$ and $N$ is represented by an area lying between $2PN$ or $2QM$, if $MN$ be sufficiently diminished; hence, it follows by reasoning similar to the above that the difference of the squares of the velocities at $A$ and $B$ is represented by twice the area $ACDB$.

56. Hence we obtain another measure for the force corresponding to the position $M$. For the increase of (velocity)$^2$ in $MN$ is represented by 2 area $PMNQ$,

$$\text{and } PM = \text{limit } \frac{PMNq}{MN},$$

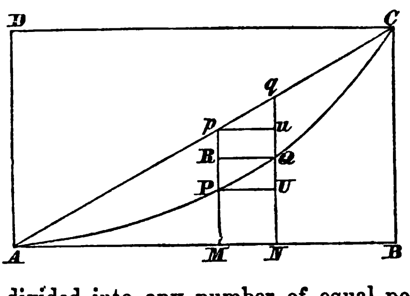
$$= \text{limit } \frac{PMNQ}{MN};$$

the accelerating effect of the force at $M$ is measured by the limit of $\frac{\text{increase of the (velocity)}^2 \text{ in } MN}{2MN}$.

*Application to the determination of the motion of a particle, under various circumstances.*

1. To find the space travelled over in a given time $t''$ by a body moving with a velocity which varies as the (time)$^2$ from the beginning of the motion.

Let $AB$ represent the time, and let $BC$ perpendicular to $AB$ represent the velocity at the end of that time, *i.e.* let $BC$ represent the space which would be described in the next unit of time, if the body instead of moving with constantly increasing velocity, were to move with uniform velocity for an unit of time from the end of the time represented by $AB$.




Let $AB$ be divided into any number of equal portions of which $MN$ is one, and let $MP$, $NQ$ represent the velocities at the end of the times represented by $AM$, $AN$.

Then since $MP : NQ : BC :: AM^2 : AN^2 : AB^2$;

$\therefore$ a parabola, whose vertex is at $A$ can be described, touching $AB$ and passing through $P$, $Q$, $C$ and the extremities of all ordinates described on $MP$.

Hence, the space described in the time represented by $AB$ is represented by the parabolic area $ABC$ or $\frac{1}{3}AB.BC$.

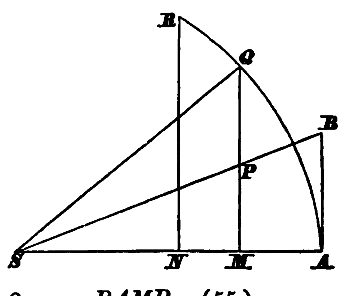
And if $p$ be the velocity at the end of 1″, $pt^2$ that at the end of $t''$; then $\frac{1}{3}pt^2.t = \frac{1}{3}pt^3$ is the space described in the time $t$.

Or, we can further illustrate the meaning of Art. 45, employing another method of representing the space.

Join $AC$, and let $pM$, $qN$ be the ordinates, and suppose the figure to revolve round $AB$, $pM$ generates a circle which $\propto pM^2 \propto AM^2$; $\therefore$ this circle may be taken to represent the velocity at the time corresponding to $AM$, and the solid generated by $pq\ NM$ represents the space described in time $MN$. The whole space is therefore represented by the cone generated by $ABC$, or $\frac{1}{3}AB.\pi BC^2$, which gives the same result as before.

2. To find the velocity acquired from rest, when a body is acted on by an attractive force whose accelerating effect varies as the distance from a fixed point.

Let $S$ be the fixed point, $A$ the point from which the motion commences, and let $AB$ perpendicular to $SA$ represent the accelerating effect of the force at $A$.




Join $SB$, and from any point $M$, let $MP$ perpendicular to $SA$ meet $SB$ in $P$;

$\therefore$ since $PM : BA :: SM : SA$,

$PM$ represents the accelerating effect of the force at $M$,

and (velocity)$^2$ at $M$ is represented by $2 \times$area $BAMP$. (55.)

Let $V$ the velocity which the force, continued uniform from $A$, would have generated in the space $\frac{1}{2}AS$; describe the circle $AQE$ with centre $S$, and produce $MP$ to $Q$.

$$\begin{aligned}(\text{velocity})^2 \text{ at } M : V^2 &:: 2 \text{ area } BAMP : AS.AB \\ &:: \Delta SAB - \Delta SMP : \Delta SAB \\ &:: SA^2 - SM^2 : SA^2 \\ &:: QM^2 : SA^2;\end{aligned}$$

$\therefore$ velocity at $M : V :: QM : SA.$

Hence, velocity at $M = V \sin QSA$,

when the body arrives at $S$ the velocity is the same, as if the force, continued constant from $A$, had acted through half that distance.

If $\mu SA$ be the measure of the accelerating effect of the force at $A$,

$$V^2 = 2\mu SA.\frac{SA}{2};$$

$$\therefore\ V = \sqrt{\mu}\, SA,$$

and the velocity at $M = \sqrt{\mu}\, QM$.

**3. Time of describing a given space from rest under the action of a force varying as the distance from a fixed point.**

Making the same construction as before;

let $t$ = time from $M$ to $N$;

$\therefore\ t \times$ velocity at $M = MN$, ultimately.

$$\begin{aligned}\text{Now, } MN : QR &:: QM : QS, \text{ ultimately} \\ &:: QM : SA \\ &:: \text{velocity at } M : V \\ &:: t \times \text{velocity at } M : tV;\end{aligned}$$

$$\therefore\ tV = QR,$$

and $V \times$ time from $A$ to $M$ = arc $AQ$;

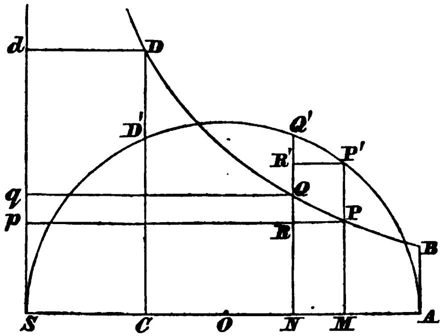
$\therefore$ the time from $A$ to $M$ is that in which a distance equal to the arc $AQ$ would be described uniformly with the velocity generated from $A$, in the space $\frac{1}{2}AS$ by the force at $A$ continued constant.

$$\text{Hence also time in } AM = \frac{\text{arc } AQ}{\sqrt{\mu}\,.\,AS}$$

$$= \frac{1}{\sqrt{\mu}} \times \text{circular measure of } QSA.$$

**4. Velocity acquired from rest by a body attracted by a force, whose accelerating effect varies inversely as the square of the distance from a fixed point.**

Let $S$ be the fixed point, $A$ the point from which the motion com mences, and let $AB$ perpendicular to $SA$ represent the accelerating effect of the force at $A$, $MP$ that at any point $M$.




Let $SA$ be divided into a large number of portions, of which $MN$ is one, and let $BPD$ be the curve which is the locus of such points as $P$, $NQ$, $CD$ ordinates at $N$ and $C$.

$Dd$, $Qq$, $Pp$, $Bb$ parallel to $AS$ meet $Sd$ perpendicular to $SA$.

Difference of (velocity)$^2$ at $M$ and $N$ is represented by

$$2PM.MN, \text{ ultimately,}$$

$$\text{but } PM : QN :: SN^2 : SM^2,$$

$$\text{and } SM : SN :: SM : SN;$$

$$\therefore \text{ rectangle } PS : \text{rectangle } QS :: SN : SM, \text{ ............. (1).}$$

$$\therefore PS : QS - PS :: SN . MN,$$

$$:: SM : MN, \text{ ultimately,}$$

$$:: PS : PN, \text{ ultimately;}$$

$$\therefore QS - PS = PN;$$

$\therefore$ (velocity)$^2$ at $N$ – (velocity)$^2$ at $M$ is represented by

$$2(QS - PS) \text{ ultimately;}$$

$\therefore$ (velocity)$^2$ generated in $AC$ is represented by $2(DS - BS)$.

If $V$ be the velocity generated by the force at $A$ continued constant through $AS$;

$$(\text{velocity})^2 \text{ at } C : V^2 :: 2(DS-BS) : 2BS,$$

$$\text{and by (1) } BS : DS :: SC : SA;$$

$$\therefore DS-BS : BS :: AC : SC$$

$$:: AC.SC : SC^2.$$

If a semicircle be described on $AS$ as diameter meeting $CD$ in $D'$.

$$(\text{velocity})^2 \text{ at } C : V^2 :: CD'^2 : SC^2;$$

$$\therefore \text{velocity at } C : V :: CD' : SC.$$

Or, if $\frac{\mu}{SC^2}$ be the accelerating effect of the force at $C$:

$$V^2 = \frac{2\mu}{SA^2}.SA = \frac{2\mu}{SA};$$

$$\text{and } (\text{velocity})^2 \text{ at } C : \frac{2\mu}{SA} :: AC : SC,$$

$$(\text{velocity})^2 \text{ at } C = \frac{2\mu}{SA}.\frac{AC}{SC}.$$

5. **Time of describing a given distance from rest under the action of the same force.**

On $SA$ as diameter describe a semicircle, center $O$, and produce $MP$, $NQ$ to the circumference in $P'$, $Q'$.

Let $t$ = time from $M$ to $N$,

$V$ = velocity generated from $A$ to $S$,

by the force at $A$ continued constant.

Then, if $P'R'$ be perpendicular to $Q'N$,

$$P'R' : P'Q' : Q'R' :: P'M :: OP' : OM, \text{ ultimately};$$

$$\therefore MN : P'Q' + Q'R' :: P'M : OP' + OM$$

$$:: P'M : SM$$

$$:: \text{velocity at } M : V$$

$$:: MN : tV \text{ ultimately};$$

$$\therefore Vt = P'Q' + Q'R';$$

$$\therefore V \times \text{time from } A \text{ to } C = \text{arc } AD' + D'C;$$

$\therefore$ time from $A$ to $C$ is that in which a space equal to arc $AD' + D'C$ would be described uniformly with the velocity generated at $S$ by the force at $A$, continued uniform,

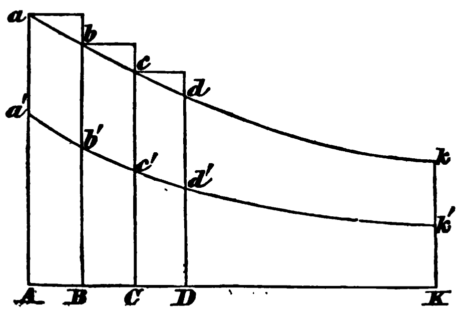
$$\text{and } V^2 = \frac{2\mu}{SA};$$

$$\therefore \text{ time to } C = \sqrt{\frac{SA}{2\mu}}\,(\text{arc } AD' + D'C),$$

$$\text{and whole time to } S = \sqrt{\frac{SA}{2\mu}}\,\pi\,\frac{SA}{2} = \frac{\pi}{\sqrt{\mu}}\left(\frac{SA}{2}\right)^{\frac{3}{2}}.$$

6. Space described by a body moving in a medium, in which the resistance varies as the velocity, and no other force acting on the body, varies as the velocity.

Let the time $AK$ be divided into equal portions $AB$, $BC$, $CD$, ...; and let $Aa'$, $Bb'$, ... be the velocities at the beginning of times, the space in time $AK$ is represented by the area $a'AKk'$.




Suppose the force of resistance to be constant throughout the intervals $AB$, $BC$, ... and equal to the amount at the commencement of each, and let $Aa$, $Bb$, ... be the measures of those forces;

$$\therefore Aa : Bb : \ldots\ldots :: Aa' : Bb' : \ldots\ldots$$

and the velocity destroyed is represented by the limit of the sum of the parallelograms $aB$, $bC$, ...... or the area $aAKk$;

$\therefore$ velocity destroyed in time $AK$

: space described :: $aAKk$ : $a'AKk$

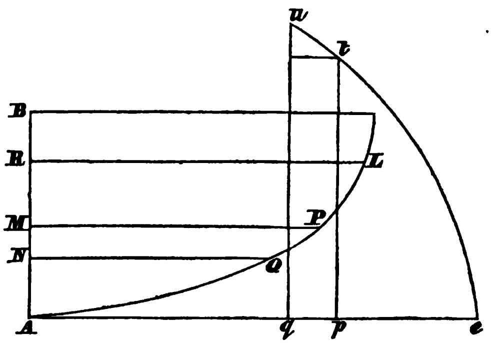
:: $Aa$ : $Aa'$ :: resistance : the velocity.

If resistance $= \mu \times$ velocity,

velocity destroyed $= \mu \times$ space.

7. A particle slides down the smooth arc of a cycloid, whose axis is vertical, to find the time of an oscillation.

Let $AB$ be the vertical axis of the cycloidal arc $APL$, $L$ the point from which the particle begins to move, $PQ$ a small arc of its path, $LR$, $PM$, $QN$ perpendicular to $AB$.




Let $v =$ velocity at $P$,

$T =$ time in falling from $B$ to $A$;

$$\therefore 2AB = gT^2,$$

$$\text{and } v^2 = 2g \cdot RM.$$

But by the properties of the cycloid, (see Appendix)

$$AL^2 = 4AB \cdot AR,$$

$$AP^2 = 4AB \cdot AM;$$

$$\therefore AL^2 - AP^2 = 4AB \cdot RM.$$

Take $Al$, $Ap$, $Aq$, on the tangent at $A$ respectively equal to $AL$, $AP$, $AQ$, and let $pt$, $qu$ perpendicular to $Al$ be ordinates to a circle whose center is $A$ and radius $Al$.

$$\therefore AL^2 - AP^2 = Al^2 - Ap^2 = pt^2.$$

$$\therefore v^2 T^2 = 2gT^2 RM;$$

$$= 4AB \, . \, RM = pt^2.$$

$\therefore$ $pt$ would be described with velocity $v$ in time $T$, and ultimately $PQ$ is described with velocity $v$;

$$\therefore \text{time in } PQ : T :: QP : pt$$

$$:: pq : pt$$

$$:: tu : At \text{ ultimately.}$$

$\therefore$ time in $PQ = T \times$ circular measure of $\angle tAu$ ultimately;

$$\therefore \text{time in } LA = T \times \frac{\pi}{2}$$

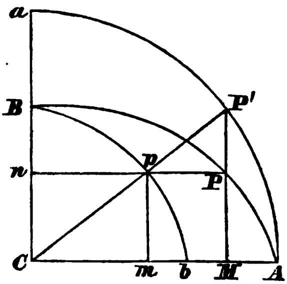
$$= \frac{\pi}{2}\sqrt{\frac{2AB}{g}};$$

$$\text{time of an oscillation} = \pi\sqrt{\frac{2AB}{g}};$$

$\therefore$ the cycloid is a tautochronous curve, i.e. the time is the same from whatever point the particle's motion commences.

8. **A particle is subject to the action of a force, whose accelerating effect varies as the distance from a fixed point, in the direction of which it acts, the particle is projected from a given point in a direction perpendicular to the direction of the force at that point, to find the path described by the particle.**

Let the force tend to $C$, $A$ the point of projection, $P$ the position of the particle at any time.




$CB$ perpendicular to $CA$, the distance in which a particle would be reduced to rest if projected from $C$ with the velocity of projection.

Describe circles $Bb$, $aA$ having the common center $C$, and draw $CpP'$ cutting the circles in $p$ and $P'$, $pn$ is perpendicular to $CB$ and $pm$, $P'M$ to $CA$.

Referring to Prob. 3, p. 81, it will be seen that particles starting respectively from rest at $A$ and with the velocity of projection at $C$, under the action of the same force, would arrive simultaneously at $M$ and $n$, since the time in both cases is proportional to the angle $P'CA$.

But the particle in the proposed problem is acted on at $P$ by a force which is represented by $PC$ whose accelerating effect parallel to $AC$ and $CB$ is represented by $MC$ and $PM$, $\therefore$ the acceleration in $AC$ is the same as that of the particle supposed to move in $AC$ from rest, and the retardation parallel to $BC$ the same as that of the particle in $CB$, projected from $C$. $\therefore$ $P$ is in the intersection of $np$ and $MP'$,

$$\begin{aligned} \text{and } PM : P'M &:: pm : P'M \\ &:: Cp : CP' \\ &:: CB : CA; \end{aligned}$$

$\therefore$ the required path of the particle is an ellipse whose semiaxes are $CA$ and $CB$.

Cor. 1. If $\mu . CP$ is the accelerating effect of the force at $P$ and $V$ the velocity of projection,

$$V^2 = \mu . CB^2.$$

$$\begin{aligned} \text{Also, area } ACP &\propto \text{area } ACP' \\ &\propto \text{angle } ACP' \\ &\propto \text{time from } A \text{ to } P, \end{aligned}$$

or the area swept out by the radius vector is proportional to the time.

Cor. 2. Also (velocity)$^2$ at $P$ = sum of the squares of the velocities of the particles at $M$ and $n$

$$= \mu P'M^2 + \mu . pn^2 = \mu CD^2,$$

if $CD$ be the semiconjugate diameter to $CP$.

## VII.

1. If the square of the velocity of a body be proportional to the space described from rest, prove that the accelerating force is constant.

2. At what point of the proof of the Lemma X is it assumed that the body starts from rest?

3. When the force is constant, how may the figure be modified, and the proposition extended?

4. What is meant by 'vis finita,' in the Lemma?

5. If a body move from rest under the action of a force, which varies as the square of the time from the beginning of the motion, shew that the velocity varies as the cube of the time, and the space described as the fourth power of the time.

6. If the velocity after a time $t$ from rest be equal to $a(2t + t^2)$, what will be the shape of the curve in the figure, and the space described in any time?

7. Find the form of the curve in Lemma X, when the force varies as the distance from a fixed point.

8. When a body moves from rest at $A$ under the action of a force which varies as the square of the distance from $S$ ($= \mu SM^2$ at $M$), the velocity at $M = \frac{\mu}{3}(SA^3 - SM^3)$.

9. If a body be acted on from rest by a repulsive force which varies as the distance from a fixed point, find the velocity when the body arrives at any position.

10. State the proposition by which Lemma X is replaced, when the body instead of starting from rest commences its motion with a given velocity.

11. A particle is placed in the line joining two centers of attracting force, the accelerating effect of which varies as the distance, find the time in which the particle oscillates.

12. Two forces reside at $S$, one attractive and whose accelerating effect on a particle varies as the distance from $S$, and the other constant and repulsive; prove that, if a particle be placed

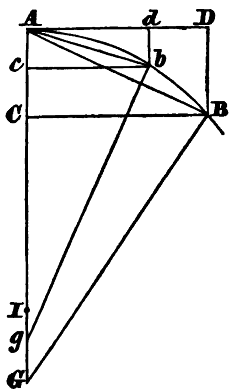
at $S$ it will move until it be brought to rest at a point which is double the distance from $S$, at which it would rest in equilibrium under the action of the forces.

## LEMMA XI.

*The vanishing subtenses of the angle of contact in all curves which have finite curvature at the point of contact, are ultimately in the duplicate ratio of the chords of the conterminous arcs.*

Case 1. Let $AB$ be the arc of a curve, $AD$ its tangent at $A$, $BD$ the subtense of the angle of contact $BAD$ perpendicular to the tangent, $AB$ the chord of the arc.

Let $AG$, $BG$ be drawn perpendicular to the tangent $AG$ and the chord $AB$ respectively, meeting in $G$; then let the points $D$, $B$, $G$ move towards the points $d$, $b$, $g$, and let $I$ be the point of ultimate intersection of the lines $BG$, $AG$, when the points $B$, $D$ move up to $A$.




It is evident that the distance $GI$ may be made less than any assigned distance by diminishing $AB$.

But since the angles $ABD$ and $GAB$ are equal, and also the right angles $BDA$, $ABG$, the triangles $ABD$, $GAB$ are similar;

$$\therefore\ BD : AB :: AB : AG,$$

$$\therefore\ BD.AG = AB^2.$$

$$\text{Similarly, } bd.Ag = Ab^2;$$

$$\therefore\ AB^2 : Ab^2 = BD.AG : bd.Ag;$$

$\therefore$ the ratio $AB^2 : Ab^2$ is a ratio compounded of the ratios of $BD : bd$, and $AG : Ag$.

But, since $GI$ may be made less than any assigned length, the ratio $AG : Ag$ may be made to differ from a ratio of equality less than by any assigned difference, $\therefore$ the ratio $AB^2 : Ab^2$ may be made to differ from the ratio $BD : bd$ less than by any assigned difference.

Hence, by Lemma I., the ultimate ratio $AB^2 : Ab^2$ is the same as the ultimate ratio of $BD : bd$. Q.E.D.

Case 2. Let now the subtenses $BD'$, $bd'$ be inclined at any given angle to the tangent; then, by similar triangles $D'BD$, $d'bd$,

$$BD' : bd' :: BD : bd,$$

and ultimately, $\quad BD : bd :: AB^2 : Ab^2;$

$\therefore$ ultimately, $\quad BD' : bd' :: AB^2 : Ab^2,$

Q.E.D.

Case 3. And although the angle $D'$ be not a given angle, if $BD'$ converges to a given point, or is drawn according to any other [fixed] law, [by which the angle $D'$ remains finite, since $BD'$ is a subtense,] still, the angles $D'$, $d'$, constructed by this law common to both, continually approach to equality and become nearer than by any assigned difference, and will be therefore ultimately equal, by Lemma I., and hence $BD'$, $bd'$, are ultimately in the same ratio as before. Q.E.D.

Cor. 1. Hence, since the tangents $AD$, $Ad$, the arcs $AB$, $Ab$, and their sines $BC$, $bc$, become ultimately equal to the chords $AB$, $Ab$; their squares also will be ultimately as the subtenses $BD$, $bd$.

Cor. 2. The squares of the same lines are also ultimately as the squares of the sagittæ of the arcs, which bisect the chords, and converge to a given point: for those sagittæ are as the subtenses $BD$, $bd$.

COR. 3. And therefore the sagittæ are in the duplicate ratio of the times in which a body describes the arcs with a given velocity.

COR. 4. The rectilinear triangles *ADB, Adb* are ultimately in the triplicate ratio of the sides *AD, Ad,* and in the sesquiplicate ratio of the sides *DB, db*; since these triangles are in the ratio compounded of *AD* : *DB* and *Ad* : *db*. So also the triangles *ABC, Abc* are ultimately in the triplicate ratio of the sides *BC, bc*. The sesquiplicate ratio is the subduplicate of the triplicate, which is compounded of the simple and the subduplicate ratios.

COR. 5. And, since *DB, db* are ultimately parallel and in the duplicate ratio of *AD, Ad,* [therefore, this being a property of a parabola, at every point at which a curve has finite curvature an arc of a parabola can be drawn which ultimately coincides with the curve ;] and the curvilinear areas *ADB, Adb* will be ultimately two thirds of the rectilinear triangles *ADB, Adb*: and the segments *AB, Ab* the third parts of the same triangles. And hence these areas and these segments will be in the triplicate ratio as well of the tangents *AD, Ad* as of the chords and arcs *AB, Ab*.

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SCHOLIUM.

But, in all these propositions, we suppose the angle of contact to be neither infinitely greater nor infinitely less than the angles of contact, which circles have with their tangents; that is, that the curvature at the point *A* is neither infinitely great nor infinitely small, in other words, that the distance *AI* is of finite magnitude.

For *DB* might be taken proportional to $AD^3$, in which case no circle could be drawn through the point *A* be-

tween the tangent $AD$ and the curve $AB$, and the angle of contact will be infinitely less than that of any circle.

And, similarly, if different curves be drawn in which $DB$ varies successively as $AD^4$, $AD^5$, $AD^6$, &c., a series of angles of contact will be presented which may be continued to an infinite number, of which each will be infinitely *less* than the preceding. And if curves be drawn in which $DB$ varies as $AD^2$, $AD^{\frac{3}{2}}$, $AD^{\frac{4}{3}}$, $AD^{\frac{5}{4}}$, $AD^{\frac{6}{5}}$, &c., another infinite series of angles of contact will be obtained, of which the first is of the same kind as in the circle, the second infinitely greater, and each infinitely *greater* than the preceding. But, moreover, between any two of these angles, an infinite series of other angles of contact can be inserted, of which each may be infinitely greater or infinitely less than any preceding; for example, if between the limits $AD^2$ and $AD^3$ there be inserted $AD^{\frac{13}{6}}$, $AD^{\frac{11}{5}}$, $AD^{\frac{9}{4}}$, $AD^{\frac{7}{3}}$, $AD^{\frac{5}{2}}$, $AD^{\frac{8}{3}}$, $AD^{\frac{11}{4}}$, $AD^{\frac{14}{5}}$, $AD^{\frac{17}{6}}$, &c. And again, between any two angles of this series there can be inserted a new series of intermediate angles differing from one another by infinite intervals. Nor does the nature of the case admit any limit.

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The propositions which have been demonstrated concerning curved lines, and the included areas, are easily applied to curved surfaces and solid contents.

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These Lemmas have been premised for the sake of escaping from the tedious demonstrations by the method of *reductio ad absurdum,* employed by the old geometers. The demonstrations are certainly rendered more concise, by the method of indivisibles; but, as

there is a harshness in the hypothesis of indivisibles, and on that account it is considered to be an imperfect geometrical method; it has been preferred to make the demonstrations of the following propositions depend on the ultimate sums and ratios of vanishing quantities and on the prime sums and ratios of nascent quantities, i. e. on the limits of sums and ratios; and therefore to premise demonstrations of those limits as concise as possible. By these demonstrations the same results are deducible as by the method of indivisibles; and we may employ the principles which have been established with greater safety. Consequently, if, in what follows, quantities should be treated of as if they consisted of particles, [indefinitely small parts,] or small curve lines should be employed as straight lines; it would not be intended to convey the idea of indivisible, but of vanishing divisible quantities, not that of sums and ratios of determinate parts, but of the limits of sums and ratios: and it must be remembered that the force of such demonstrations rests on the method exhibited in the preceding Lemmas.

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An objection is made, that there can be no ultimate proportion of vanishing quantities; inasmuch as before they have vanished the proportion is not ultimate, and when they have vanished, it does not exist. But by the same argument it could be maintained that there could be no ultimate velocity of a body arriving at a certain position, at which its motion ceases; for that this velocity, before the body arrives at that position, is not the ultimate velocity; and that, when it arrives there, there is no velocity. And the answer is easy: that, by the ultimate velocity is to be understood that,

when the body is moving, neither before it reaches the last position, and the motion ceases, nor after it has reached it, but at the instant at which it arrives; i. e. the very velocity *with which* it arrives at the last position, and *with which* the motion ceases.

And similarly, by the ultimate ratio of vanishing quantities is to be understood the ratio of the quantities, not before they vanish, nor after, but *with which* they vanish. Likewise also, the prime ratio of nascent quantities is the ratio *with which* they begin to exist. And a prime or ultimate sum is that *with which* it begins to be increased or ceases to be diminished.

There is a limit, which the velocity can attain at the end of the motion, but cannot surpass. This is the ultimate velocity. And the like can be stated of the limit of all quantities and proportions commencing or ceasing to exist. And since this limit is certain and definite, to determine it is strictly a geometrical problem. And all geometrical propositions may be legitimately employed in determining and demonstrating other propositions which are themselves geometrical.

It may also be argued, that if the ultimate ratios of vanishing quantities be given, the ultimate magnitudes will also be given, and thus every quantity will consist of indivisibles, contrary to what Euclid has demonstrated of incommensurable quantities, in his tenth book of the Elements.

But this objection rests on a false hypothesis. Those ultimate ratios with which quantities vanish, are not actually ratios of ultimate quantities, but limits to which the ratios of quantities decreasing without limit are continually approaching; and which they can approach nearer than by *any* given difference, but which they

can never surpass, nor reach before the quantities are indefinitely diminished.

The argument will be understood more clearly in the case of infinitely great quantities. If two quantities, of which the difference is given, be increased infinitely, their ultimate ratio will be given, namely, a ratio of equality, yet in this case the ultimate or greatest quantities of which that is the ratio will not be given.

In what follows, therefore, if at any time for the sake of facility of conception, the expressions *indefinitely small*, or *vanishing*, or *ultimate* be used concerning quantities, care must be taken to understand thereby quantities determinate in magnitude, but to conceive them in all cases quantities to be diminished without limit.

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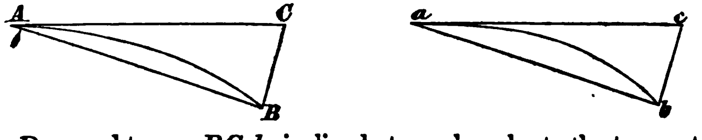
*Curvature of Curves.*

57. The curvature of a curve at any point is greater or less as the amount of deflection from the tangent at that point, in the immediate neighbourhood of the point, is greater or less.

Two curves have the same curvature at two points, taken one in each, if they have the same deflection from the tangents at those points at equal distances from the points of contact, in the immediate neighbourhood of those points.

58. An exact geometrical test of equality of curvature may be obtained as follows:—

If $AB$, $ab$ be two curves which have the same curvature at $A$, $a$ respectively, draw the tangents $AC$, $ac$ and take $AC = ac$.




Draw subtenses $BC$, $bc$ inclined at equal angles to the tangents.

If $BC$ and $bc$ were equal for all equal values of $AC$, $ac$ the curves would be equal and similar. If $BC : bc$ be ultimately a ratio of equality when $AC$, $aC$ are taken indefinitely small, the curves have the same deflection from the tangents in the immediate neighbourhood of $A$, $a$, or the curves have the same curvature at those points.

If the chords $AB$, $ab$ be drawn, it is an immediate consequence that the ultimate ratio of the angles $BAC$, $bac$ is a ratio of equality. These angles are called the angles of contact.

Hence, curves have the same curvature at two points, taken one in each, if, equal tangents being drawn at those points, and subtenses inclined at any equal angles to the tangents, the limiting ratio of the subtenses is a ratio of equality, or if the limiting ratio of the angles of contact be a ratio of equality.

59. The curvature of one curve is infinitely greater or infinitely less than that of another if the limiting ratio of the subtense of the first to that of the second be infinitely great or infinitely small.

60. The ratio of the curvature of one curve to that of another at two points, or of the curvature of the same curve at two different points, is the limiting ratio of the subtenses drawn to the extremities of equal tangents and inclined at equal angles to the tangents.

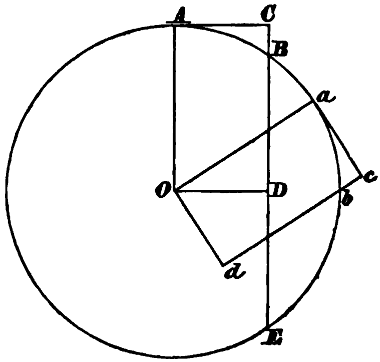
61. The curvature of a curve is said to be finite, at any point, when the ratio of the curvature at that point to that of a circle is finite.

62. *The curvature of a circle is the same at every point.*

Let $A$, $a$ be any two points on a circle, $AC$, $ac$ equal tangents at $A$, $a$, $CB$, $cb$ subtenses perpendicular to the tangents, $OD$, $Od$ perpendicular to the subtenses produced;

$\therefore$ $CD = cd$, each being equal to the radius, and $BD = bd$;

$\therefore BC = bc$, and ultimately, when the arcs are indefinitely diminished, $BC : bc$ is a ratio of equality;




$\therefore$ the circle has the same curvature at $A$, $a$ any two points.

63. *In different circles the curvature varies inversely as the radii.*

In the last figure, produce $CB$ to the circumference in $E$. Then, $AC^2 = CB \cdot CE$, and if $A'C' = AC$ be a tangent to another circle, and the same construction be made

$$A'C'^2 = C'B' \cdot C'E';$$

$$\therefore CB \cdot CE = C'B' \cdot C'E';$$

$$\text{and } CB : C'B' :: C'E' : CE;$$

and, ultimately, when $AC$, $A'C'$ are indefinitely diminished,

$$CE = 2AO;$$

$$\therefore CB : C'B' :: A'O' : AO, \text{ ultimately,}$$

or the curvatures are inversely proportional to the radii.

*Measure of Curvature.*

64. The curvature of a circle is the same at every point; the curvature of different circles varies inversely as the diameters

of the circles; and a circle can be constructed of any degree of finite curvature by varying the magnitude of the diameter.

Hence, a circle can always be found, whose curvature at any point is equal to that of a curve at a fixed point.

The curvature of a curve at any point is therefore completely determined, when the diameter of the circle is found, which has the same curvature as the curve at the given point.

The diameter of the circle, which has the same curvature as the curve at a given point, is called *the diameter of curvature of the curve at that point.*

The chord of the circle, drawn in any direction, is called *the chord of curvature in that direction.*

The circle itself is called *the circle of curvature,* and is the circle which has the same tangent as the curve at any point, and also the same curvature.

65. Any other curve might have been chosen as the standard measure of finite curvature, but, since no curve but the circle has the same curvature at every point, it would then have been necessary, after selecting the curve, to specify the point at which the curvature might form the measure of curvature.

Thus, if the standard curve were a parabola, we must choose the curvature of the parabola at the vertex or at the extremity of the latus rectum or at some determinate point, by which to obtain the measure.

The inconvenience is obvious.

### *General Properties of the Circle of Curvature.*

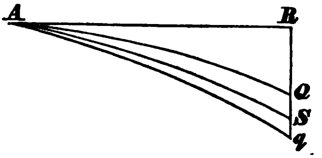
66. If a circle be drawn touching a curve at a given point, and cutting it at a second point, as the second point approaches indefinitely near the point of contact, the circle assumes a limiting magnitude, and evidently satisfies the condition that it has the same curvature as the curve at that point.

67. As a tangent at any point is the limiting position of a side of a polygon terminated in that point, and inscribed in the curve, when the number of sides is increased indefinitely: so the circle of curvature at any point is the limiting circle which passes through the extremities of two consecutive sides of the polygon either terminated in that point or commencing from that point.

68. *No circle can be drawn whose circumference lies between a curve and its circle of curvature at any point, near that point.*

For, let $AQ$ be the arc of the curve, $Aq$ of the circle of curvature; and let, if possible, another circle be drawn, of which the arc $AS$ lies between the curve and circle.

Draw the tangent $AR$ and let $RQ$, the subtense perpendicular to the tangent, cut the circles in $q$, $S$.




Then $SR : qR$ is ultimately the ratio of the diameters of the circles:

$\therefore$ $SR$ is ultimately unequal to $qR$;

but since $qR$ and $QR$ are ultimately in a ratio of equality, $SR$ which is intermediate in magnitude is ultimately equal to either, which is absurd.

$\therefore$ no circle, &c.

This proposition corresponds to Euclid, III, Prop. XVI.

69. *The circle of curvature generally cuts the curve.*

For the curvature of the curve at different points taken along the curve continually increases or continually diminishes, until it arrives at a maximum or minimum value.

If therefore the circle of curvature be drawn at any point, on the side on which the curvature is increasing, as we proceed

from the point, the curve lies within the circle, and on the other side, on which the curvature is diminishing, the curve lies without the circle; which proves the proposition in the general position of the point.

For the particular case, in which the point is at a position of maximum or minimum curvature, as at the extremities of the axes of an ellipse, if the curvature be a maximum the curvature at adjacent points on either side is less than that of the circle of curvature at the point under consideration, therefore the circle lies entirely within the curve on both sides near the point of maximum curvature; and similarly, it lies without the curve at points of minimum curvature.

70. To illustrate this by reference to the inscribed polygon.

If, in a curve, equal chords $AB$, $BC$, $CD$, $DE$, ... be placed in order, generally the angles $ABC$, $BCD$, $CDE$,... increase or decrease, commencing from any point, which property of the polygon has the property corresponding to it in the curvilinear limit, when the chords are diminished indefinitely, that the curvature decreases or increases continually.

Suppose the angles are increasing from $B$, in the circle described about $BCD$, let $BA'$, $DE'$ be placed equal to $BC$ or $CD$.

Then, $BA'$ and $DE'$ lie on opposite sides of the perimeter of the polygon, whence, if we proceed to the limit, the circle of curvature at a point in the middle of increasing curvature cuts the curve.

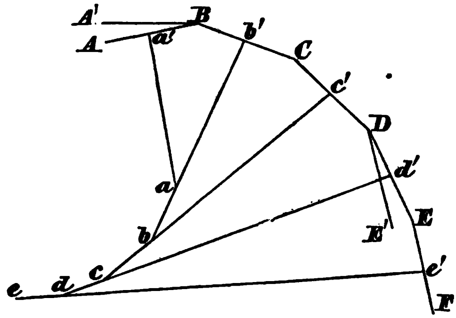
If the angles $ABC$ and $DEF$ be each less than the angles $BCD$ $CDE$, supposed equal, the curvature decreases and then increases, and the circle about $BCD$ passes through $E$, and $BA$, $EF$ lie within the circle, and proceeding to the limit, the circle of curvature lies without the curve, near the point of minimum curvature.

*Evolute of a Curve.*

71. If the circles of curvature be drawn at every point of a curve, the centers of those circles lie in a curve which is called the *evolute* of the proposed curve.

*Properties of the Evolute.*

72. Let $ABCDE$ be any polygon, and let $a'a$, $b'b$, $c'c$, $d'd$ be drawn perpendicular to the sides from the middle points $a'$, $b'$, &c., these intersect in the angular points $abcd$... of another polygon.




If a string were wrapped round $a'abcd$... the extremity $a'$ would as the string was unwrapped pass through the points $a'$ $b'$ $c'$ $d'$.

Let now the number of sides of the polygon be increased and the magnitude diminished indefinitely.

The points $a'$ $b'$ $c'$... are ultimately in the curve which is the limit of the polygon, and since $a$, $b$, $c$, ... are the centers of the circles described about $ABC$, $BCD$,... $a$, $b$, $c$,... are ultimately the centers of the circles of curvature at $a'$ $b'$ $c'$..., and the curve which is the limit of the polygon $abcd$... is the evolute of the curve $a'$ $b'$ $c'$..., and the property proved for the polygons is true for the limits of the polygons, therefore the extremity of the string wrapped round the evolute traces the curve of which it is the evolute. This property gives rise to the name of evolute.

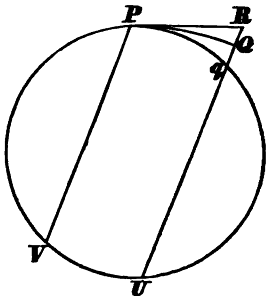
Also $b'$ $b$ is ultimately the tangent to the evolute and is perpendicular to $BC$ which is ultimately the tangent to the curve $a'$ $b'$ $c'$..., therefore the tangent to the evolute is a normal to the curve.

The curves formed by the unwrapping of the string from the evolute are called involutes.

*Propositions on Diameters and Chords of Curvature.*

73. *If a subtense be drawn from the extremity of an arc of finite curvature, in any direction, the chord of curvature parallel to that direction is the limit of the third proportional to the subtense and the arc.*

Let $PQ$, $Pq$ be arcs of a curve and its circle of curvature at $P$, $PR$ the common tangent, $RQq$ the direction of a common subtense, meeting the circle in $U$.




$PV$ parallel to $RQ$;

$\therefore$ since $Rq \,.\, RU = PR^2$,

$RU$ is the third proportional to $PR$ and $Rq$.

But, ultimately, when $PQ$ is indefinitely diminished $RU = PV$, $PR = PQ$, (Lemma VII); and $Rq = QR$ by the properties of the circle of curvature;

$\therefore$ $PV$ is the limit of the third proportional to $QR$ and $PQ$.

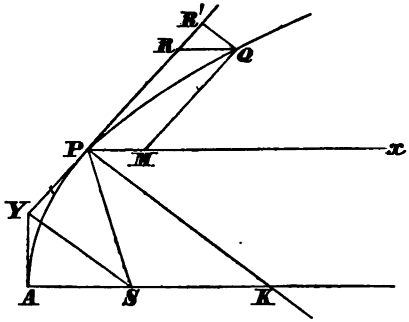
Cor. *The diameter of curvature is the limit of the third proportional to the arc and the subtense perpendicular to the tangent.*

74. *The chords of curvature at any point of a parabola drawn through the focus, and in the direction of a diameter are equal to four times the focal distance of that point.*

Let $AP$ be a parabola, $P$ any point, $RQ$ a subtense parallel to the diameter $PMx$, $QM$ the ordinate at $Q$, $S$ the focus. Then, by the properties of the parabola,

$$QM^2 = 4SP \,.\, PM;$$

$\therefore$ $4SP$ is a third proportional to $QM$ and $PM$, i.e. to $PR$ and $RQ$;




$\therefore$ $4SP$ is the limit of the third proportional to the arc $PQ$ and the subtense $QR$, and is therefore equal to the chord of curvature at $P$ in direction of the diameter.

And since $PS$, $PM$ are equally inclined to the tangents at $P$, the chords in those directions are equal; therefore, the chord of curvature through $S$ is four times the focal distance $SP$.

75. *One fourth of the diameter of curvature at any point of a parabola is a third proportional to the focal distance of that point, and the perpendicular from the focus on the tangent at that point.*

For, draw $SY$, $QR'$ perpendicular to $PR$, and let $II$ be the diameter of curvature at $P$.

$$\text{Then } PI : PQ :: PQ : QR' \text{ ultimately;}$$
$$\therefore PI : PR :: PR : QR' \text{ ultimately.}$$
$$\text{But, } PR : 4SP :: QR : PR;$$
$$\therefore PI : 4SP :: QR : QR' \text{ ultimately,}$$
$$:: SP : SY.$$

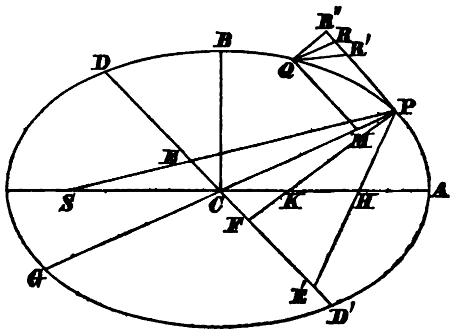
Since the triangles $SYP$, $QR'R$ are similar;

$$\therefore 4SP^2 = PI \, . \, SY;$$

$$\therefore \tfrac{1}{4} PI \text{ is a third proportional to } SP \text{ and } SY.$$

76. *The chord of curvature at any point of an ellipse, drawn through the center of the ellipse, is a third proportional to the diameter through that point and the diameter conjugate to it.*

Let $P$ be any point in an ellipse, $PCQ$ the diameter, $DCD'$ conjugate to it, $Q$ any point near $P$, $QR$ a subtense




parallel to $CP$, $QM$ an ordinate parallel to $DC$, $PV$ the chord of curvature drawn through $C$.

$$\text{Then, } PV \, . \, QR = PQ^2 = QM^2, \text{ ultimately,}$$
$$\text{and } QM^2 : PM \, . \, MG :: CD^2 : CP^2;$$
$$\therefore PV \, . \, QR : QR \, . \, MG :: CD^2 : CP^2, \text{ ultimately.}$$
$$\therefore PV : 2CP :: CD^2 : CP^2, \text{ ultimately:}$$
$$\therefore PV \, . \, CP : 2CP^2 :: CD^2 : CP^2,$$
$$\text{and } PV \, . \, CP = 2CD^2;$$
$$\text{or } PV : 2CD :: 2CD : 2CP;$$

or $PV$ is a third proportional to $PG$ and $DCD'$.

77. *The chord of curvature at any point through the focus is a third proportional to the major axis, and the diameter parallel to the tangents at that point.*

Draw the focal distance $SP$ cutting the diameter $DCD'$ in $E$, and let $PV'$ be the chord of curvature through $S$.

Then $PV' : PV :: QR : QR'$, ultimately,

$$:: CP : PE, \text{ by similar triangle;}$$

$$\therefore PV'.PE = PV.CP = 2CD^2;$$

$\therefore$ $PV'$ is a third proportional to $2PE$ and $DCD'$, and $2PE$ is equal to the major axis; and similarly for the other focus $H$.

78. *The diameter of curvature at any point, is a third proportional to twice the perpendicular from the point on the diameter parallel to the tangent and that diameter.*

Draw $QR''$ perpendicular to the tangent, and $PF$ perpendicular to $DCD'$, and let $PI$ be the diameter of curvature.

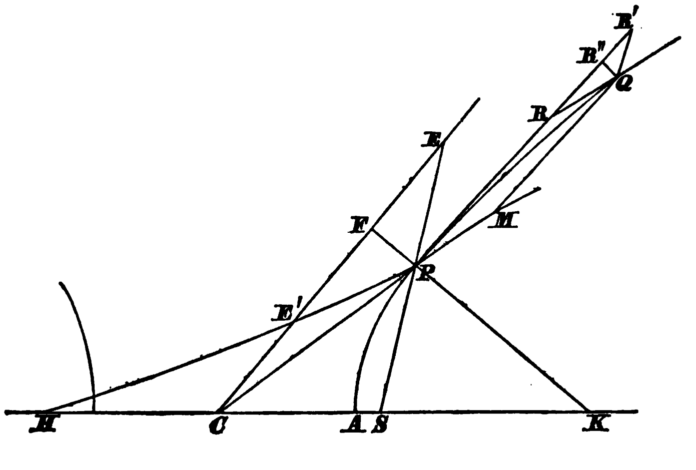
$$PI : PV :: QR : QR'',$$

$$:: CP : PF;$$

$$\therefore PI.PF = PV.CP = 2CD^2;$$

$\therefore$ $PI$ is a third proportional to $2PF$ and $DCD'$.

79. The propositions concerning the chords and diameter of curvature of an ellipse may be proved in the same words for the hyperbola, employing the following figure.



80. *The radius of curvature at any point of a conic section is to the normal in the duplicate ratio of the normal to the semi-latus rectum.*

Let $PK$ be the normal, $PO$ the radius of curvature, $L$ the latus rectum.

I. For the parabola,

$$PO : 2SP :: SP : SY,$$
$$:: SY : SA,$$
$$\text{and } PK = 2SY, \text{ or } PK^2 = 4SP . SA;$$
$$\therefore PO : 2SY :: SP : SA,$$
$$:: 4SP . SA : \left(\frac{L}{2}\right)^2;$$
$$\therefore PO : PK :: PK^2 : \left(\frac{L}{2}\right)^2.$$

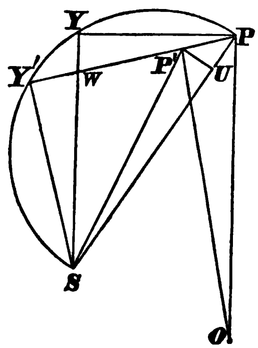
II. For the ellipse or hyperbola,

$$PO . PF = CD^2, \text{ and } PK . PF = BC^2;$$
$$\therefore PO : PK :: CD^2 : BC^2,$$
$$:: AC^2 : PF^2;$$
$$\text{and } AC : PF :: AC . PK : PF . PK = BC^2 = \frac{L}{2} . AC,$$
$$:: PK : \frac{L}{2};$$
$$\therefore PO : PK :: PK^2 : \left(\frac{L}{2}\right)^2, \quad \text{Q.E.D.}$$

81. *To find the radius of curvature of a curve defined by the relation between the radius vector and the perpendicular from the pole on the tangent.*

Let $QPP'$ be a polygon inscribed in a curve, $SY$, $SY'$ perpendicular on the sides $QP$, $PP'$; $PO$, $P'O$ perpendicular to the

same sides intersecting in $O$, $P'U$ perpendicular $SP$, and let $SY$, $PY'$ intersect in $W$.




Describing a semicircle $PYY'S$ on $SP$

$\angle YPW = \angle YSY' = \angle POP'$,

and $\angle WYP = \angle OP'P$;

therefore the triangles are similar.

$\therefore PO : PP' :: PW : YW$;

also, $PP' : SP :: PU : PY'$,

by similar triangles $P'UP'$, $SY'P$;

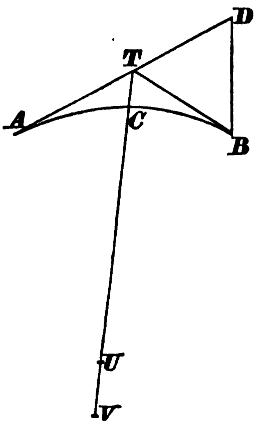
therefore, ultimately, since $PW = PY'$ ultimately,

$$PO : SP :: PU : YW$$

$$:: SP \sim SP' : SY \sim SY', \text{ ultimately,}$$

and chord of curvature at $P$ : $2SY$ :: $SP \sim SP'$ : $SY \sim SY'$, ultimately.

82. *Two tangents* AT, BT *are drawn at the extremities of an arc* AB, *to prove that* AT *is ultimately equal to* BT, *when* AB *is indefinitely diminished.*




Draw $TCUV$ in any direction making a finite angle with the tangents, and meeting the circles of curvature at $A$ and $B$ in $UV$. Then since the circle of curvature at $A$ is the limit of the circle which passes through $C$ and has the tangent $AT$, we have ultimately,

$$TA^2 : TB^2 :: TC.TU : TC.TV,$$

and $TU = TV$, ultimately;

$\therefore TA = TB$, ultimately.

COR. If the subtense $BD$ be drawn

$$AT + TB = AB = AD, \text{ ultimately};$$

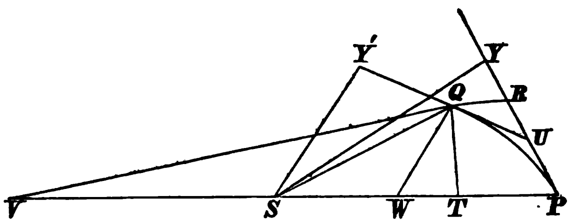
therefore, $T$ is ultimately the point of bisection of $AD$.

*Illustrations.*

1. The chord of curvature at any point of an equiangular spiral, which is drawn through the focus is equal to twice the focal distance of the point.

Let $PQ$ be an arc of the equiangular spiral, $PR$, $UQY'$ tangents at $P$ and $Q$, $QR$ a subtense parallel to $SP$,

$$\angle SQY' = \angle SPR$$




by the properties of the curve,

$\therefore$ $SQU$, $SPU$ are together equal to two right angles;

$$\therefore \angle PSQ = \angle QUR.$$

Draw $QW$ making

$$\angle QWP = \angle RPW;$$

$\therefore$ triangles $SWQ$, $URQ$ are similar,

and $SQ : QW :: QU : QR$;

$\therefore$ $2SQ : QW :: 2QU : QR$,

and $QW = PR$, also $2QU = PR$ ultimately by the last proposition,

$\therefore$ $2SP : PR :: PR : QR$ ultimately;

$\therefore$ $2SP$ is the chord of curvature at $P$ through $S$.

*Aliter.*

Since $SY : SY' :: SP : SP'$,

$SY : SP :: SY \sim SY' : SP \sim SP'$,

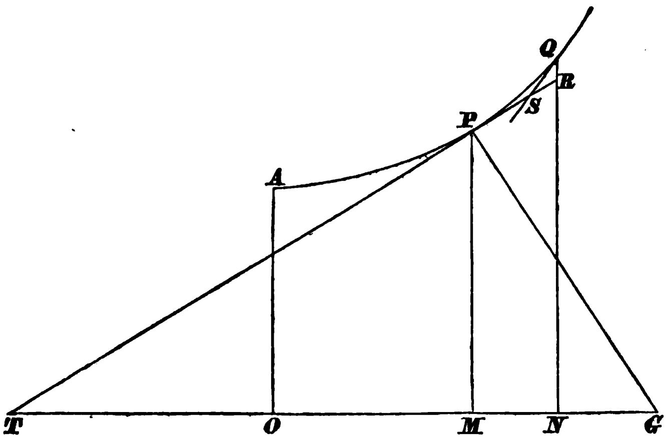
$:: 2SY$ : chord of curvature at $P$, by Art 81;

$\therefore$ chord of curvature at $P$ through $S = 2SP$.

2. **Radius and vertical chord of curvature of a catenary.**

Let $PQ$ be a small arc of a catenary $RSPT$, $QS$ tangents at$^n$ $P$ and $Q$, $PM$, $QN$ ordinates.

$TOM$ the directrix.




By the triangle of forces $QSR$ (see Appendix).

Tension at $P$ : weight of $PQ$ :: $SR$ : $QR$;

$\therefore$ $PM$ : $PQ$ :: $SR$ : $QR$,

:: $\frac{1}{2}PQ$ : $QR$ ultimately;

$\therefore$ $2PM$ is the limit of the third proportional to $QR$ and $PQ$ and is $\therefore$ the vertical chord of curvature.

Hence, the radius of curvature = $PG$ the normal.

Also, $PG$ : $PM$ :: $PT$ : $TM$,

:: tension at $P$ : tension at $A$,

:: $PM$ : $AO$;

$\therefore$ the radius of curvature at $P$ is a third proportional to $PM$ and $AO$.

3. **To find the chord of curvature at any point of the cardioid, through the focus.**

Reverting to the construction used in the example, Art. 39, it is easily seen that $SY$ being perpendicular to $PT$, the triangle

$$PSY,\ pBm, \text{ and } CBp \text{ are similar;}$$

$$\therefore SY : SP :: Bm : Bp,$$

$$:: Bp : BC;$$

$$\therefore SY^2 : SP^2 :: SP : BC,$$

and by Article 81, we have ultimately chord of curvature

$$: 2SY :: SP \sim SP' : SY \sim SY',$$

$$\text{and } (SY^2 - SY'^2)\,BC = SP^3 - SP'^3;$$

$$\therefore \text{ultimately } SP - SP' : SY - SY' :: 2SY.BC : 3SP^2,$$

$$:: 2SP : 3SY;$$

$$\therefore \text{chord of curvature through } S = \frac{4}{3}.SP.$$

## VIII.

1. Shew that the diameter of curvature at any point in a conic section varies as the cube of the normal.

2. Prove that the focal distance of the point in the parabola at which the curvature is $\frac{1}{8}^{\text{th}}$ of that at the vertex is equal to the latus rectum.

3. Prove that the diameter of curvature at the vertex of the major axis of an ellipse is equal to the latus rectum: and shew that the ratio of the curvatures at the extremities of the axes is that of the cubes of the axes.

4. Apply the property that the radius of curvature at any point of an ellipse is to the normal in the duplicate ratio of the normal to the semi-latus rectum, to shew that the radius of curvature at the extremity of the major axis is equal to the semi-latus rectum.

5. Find for what point of an ellipse the circle of curvature passes through the other extremity of the diameter at that point, shew that the distance of this point from the center is the side of the square of which $AB$ is the diagonal.

6. If $BC$ be the chord of an arc $BAC$ of continued curvature, $A$, $D$ the middle points of the arc and chord, does $AD$ pass through the center of curvature ultimately, when the arc is indefinitely diminished?

7. In a rectangular hyperbola, the diameter of curvature at any point, and the chords of curvature through the focus and center are in geometrical progression.

8. $A$, $B$, $C$ are three points in a curve of finite curvature: when $A$ and $C$ move up to $B$, and ultimately coincide with it, the circle circumscribing the triangle formed by the tangents at $A$, $B$, and $C$ will ultimately cut the normal at $B$ in a point which is at a distance from $B$ equal to half the radius of curvature there, and the triangle formed by those tangents is ultimately half of the triangle $ABC$.

9. Prove that the chord of curvature through the vertex $A$ of a parabola is $\frac{4PY^2}{AP}$, $Y$ being the intersection of the tangents at $P$ and $A$.

10. Shew that the sum of the chords of curvature through a focus of an ellipse at the extremities of conjugate diameters is constant. Also, that the sum of the radii of curvature raised to the power $\frac{2}{3}$ is constant.

11. If $PSp$ be a focal chord of an ellipse, $PT$, $pT$ tangents at $P$ and $p$, shew that the curvatures at $P$ and $p$ are as the cubes of $pT$ and $PT$.

12. Prove that at a point $P$ in an ellipse for which the minor axis is a mean proportional between the radius of curvature and the normal, $CP = AC - BC$. Shew that this is impossible unless $AC \geqq 2BC$.

13. If the radius of curvature is twice the normal

$$CP = CS.$$

If moreover $AC = 2BC$, $CP = 3PM$.

14. Shew that the evolute of an equiangular spiral is a similar spiral, and also that the extremities of the diameter of curvature traces out a similar spiral.

15. In an ellipse the circle described on the semi-major axis is the circle of curvature at the vertex, shew that if $SL$ the semi-latus rectum be produced to the auxiliary circle in $U$, $SU = SC$.

16. If $x$, $y$ be the co-ordinates of a point $P$ of a curve $OP$ passing through the origin $O$, the radius of curvature at $O$

$$= \frac{1}{2} \text{ limit } \frac{x^2 + y^2}{x \sin \alpha \sim y \cos \alpha},$$

$\alpha$ being the inclination of the tangent at $O$ to the line of abscissæ.

Hence shew that if the equation of a curve be

$$y^2 + 2ay - 2ax = 0,$$

the radius of curvature at the origin is $2\sqrt{2} \cdot a$.

17. Prove that the chord of curvature at any point of the Lemniscate drawn through the focus is two thirds of the radius vector.

---

*Observations on the Lemma.*

83. In the proof of Lemma XI, $AI$ is the limit of the third proportional to $AB$ and $BD$, hence it is the diameter of curvature to the curve at $A$.

84. For an example of a law according to which in Case 3, the directions of the subtenses may be determined, we may suppose that they always pass through a point given in position, at a finite distance from $A$; or, that they always touch a given curve, but it must be observed that the case in which they touch a curve, which has the same tangent $AD$ at $A$, is excluded, since in this case the angles $D'$, $d'$ do not remain finite, a property required in the name subtense.

85. Cor. 2. If the sagitta $EF$ bisect $AB$ in $E$ and be produced to the tangent in $G$, in fig. page 89,

$$\begin{aligned} FG : BD &:: AE^2 : AB^2 \text{ ultimately,} \\ &:: 1 : 4, \\ \text{and } BD : GE &:: AB : AE \\ &:: 2 : 1; \\ \therefore FG : GE &:: 1 : 2, \\ \therefore FE &= FG \text{ ultimately,} \\ \therefore FE : BD &:: 1 : 4 \text{ ultimately,} \end{aligned}$$

and the sagitta of an arc which bisects the chord varies as the subtense at the extremity of the arc parallel to the sagitta.

86. Cor. 3. The parabola mentioned in this corollary is a parabola of curvature at that point; and, since $DB$ may be taken in any given direction, the proposition

$$BD : bd :: AD^2 : Ad^2$$

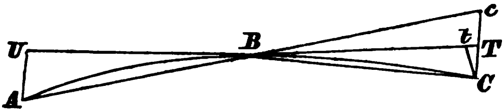
remaining, therefore the line through $A$ drawn in that direction is the corresponding diameter of the parabola of curvature.

Hence, the axis of the parabola may be taken in any direction.

If the subtenses be perpendicular to the tangent, the parabola of curvature is the parabola whose curvature at the vertex determines the curvature of the curve, and the axis is perpendicular to the tangent, and if $4AU$ (fig. page 116) be the third proportional to the arc and subtense, the limiting position of $U$ is the focus of the parabola.

By means of this corollary, the proposition alluded to under Lemma IX. Art. 43, is established; viz. that the proportion of the areas which takes place of the duplicate ratio, obtained in that Lemma, is the triplicate ratio of the same lines, when instead of a finite angle made with the tangent, the cutting line coincides with the tangent.

87. In order to shew the danger of falling into an error by a careless employment of the propositions proved in the first section, the following fallacious proof may be noticed of the proposition, that if $BT$ be a tangent to a curve $BC$ of finite curvature at the point $B$, and $BT$ be taken equal to $BC$ and $CT$ joined, $CT$ is ultimately parallel to the normal at $B$.




For, joining $BC$, $BT : CB$ is ultimately a ratio of equality by Lemma VII; therefore $CBT$ being an isosceles triangle ultimately, $CT$ is perpendicular to the line bisecting the angle $CBT$, and therefore to the tangent $BT$, since $BT$ and $BC$ ultimately coincide.

Lemma VII only allows us to assert that $BT$ and $BC$ differ by a quantity $Tt$ which vanishes compared with either of them, therefore does not distinguish between $CT$ and $Ct$. Now by Lemma XI, $CT \propto BC^2$,

$$\text{and } Tt \text{ may } \propto BC^2;$$

$\therefore$ $Tt : CT$ may be a finite ratio, or $CT$ ultimately inclined at any finite angle to $BT$, at least as far as the reasoning given is concerned.

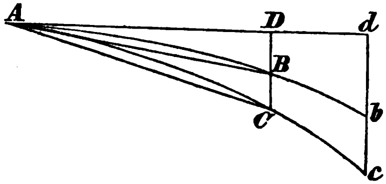
88. *If* BT *be a tangent at* B, AB, BC, *equal chords of a curve of finite curvature, drawn from* B, *and* AB *be produced to* c, *making* Bc = AB, *and* Cc *be joined meeting* BT *in* T, Cc *is ultimately* = 2 CT, *when the arcs* AB, CB *are diminished indefinitely.*

For, if $AU$ be drawn parallel to $CT$,

$$CT : AU :: AB^2 : BC^2 \text{ ultimately,}$$

$$\therefore\ CT = AU \text{ ultimately},$$
$$\therefore\ AC \text{ is ultimately parallel to } BT, \text{ and } Ac = 2AB;$$
$$\therefore\ cC = 2CT \text{ ultimately}.$$

89. Scholium. Let $AB$, $AC$ be two curves, having a common tangent $AD$ at $A$, and let subtenses $DB$, $DC$ of the




angles of contact be drawn from $D$ at any point in the tangent in the same direction, and let $BD \propto AD^m$, $CD \propto AD^n$ in the curves $AB$, $AC$ respectively.

Draw $dbc$ a common ordinate from a fixed point $d$, parallel to $DBC$.

$$\text{Then } AD^m : Ad^m :: BD : bd,$$
$$AD^n : Ad^n :: CD : cd,$$

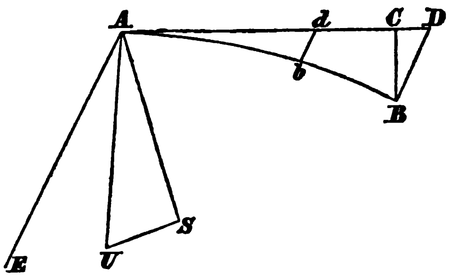
and if $m$ be greater than $n$, $= n + r$ suppose,

$$AD^n.AD^r : Ad^n.Ad^r :: BD : bd;$$
$$\therefore\ CD.AD^r : cd.Ad^r :: BD : bd$$
$$:: BD.AD^r : bd.AD^r;$$
$$\therefore\ CD : BD :: cd.Ad^r :: bd.AD^r,$$

and since $b$, $c$, $d$ are fixed, and $AD$ vanishes in the limit, therefore $CD$ is infinitely greater than $BD$, and since the angle of contact, $BAD$, $CAD$ are ultimately proportional to $CD$, $BD$, it follows that, if in two curves the subtenses vary according to different powers of the arcs or tangents, the angle of contact of that curve in which the index of the power is the least is infinitely greater than the angle of contact of the other.

*Illustrations.*

1. To construct for the axis and focus of the parabola of curvature for any direction of the parallel subtenses.




Let $AB$ be the curve of finite curvature, $BD$, $bd$ subtenses, $AE$ parallel to them.

Draw $AU$ perpendicular to $AD$, and $AS$ making angle $UAS=UAE$, and since $AE$ is a diameter of the parabola, $AS$ is in the direction of the focus.

and if $4AS$ be a third proportional to $BD$ and $AD$;

$$\therefore AD^2 = 4AS.BD,$$

and since $AD^2 : Ad^2 :: BD : bd$ ultimately;

$\therefore$ the limiting position of $S$ is the focus of the parabola.

2. To find the locus of $S$ when $BD$ is inclined at different angles to $AD$.

Let $BC$ be perpendicular to $AD$, and $AU$ be chosen so that

$$4AU : AC :: AC : BC,$$

the limiting position of $U$ is the focus of the parabola whose curvature at the vertex is the same as that of the curve at $A$,

and $AD : 4AS :: BD : AD$ ultimately:

$\therefore$, since $AD = AC$ ultimately,

$$AU : AS :: BD : BC,$$

and $\angle SAU = \angle DBC$,

join $SU$, $\therefore$ the triangles $SAU$, $DBC$ are similar,

and $\angle ASU = \angle BCD =$ a right angle;

$\therefore$ the locus of $S$ is a circle on $AU$ as diameter.

3. $ABC$ (fig. Art. 87.) is an arc of finite curvature, and is divided so that $AB : AC :: m : n$, a constant ratio.

Join $AB$, $AC$, $BC$, and shew that, ultimately,

$$\triangle ABC : \text{segment } ABC :: 3 : \left(\sqrt{\frac{m}{n}} + \sqrt{\frac{n}{m}}\right)^2.$$

For by Cor. 5. Lemma XI.

$$\text{seg } AB : \text{seg } ABC :: AB^3 : ABC^3$$
$$:: m^3 : (m+n)^3$$
$$\text{seg } BC : \text{seg } ABC :: n^3 : (m+n)^3;$$
$$\therefore \text{seg } AB + \text{seg } BC : \text{seg } ABC :: m^3 + n^3 : (m+n)^3,$$
$$\text{and } \triangle ABC = \text{seg } ABC - \text{seg } AB - \text{seg } BC;$$
$$\therefore \triangle ABC : \text{seg } ABC :: 3(m^2n + mn^2) : (m+n)^3$$
$$:: 3 : \frac{(m+n)^2}{mn},$$
$$:: 3 : \left(\sqrt{\frac{m}{n}} + \sqrt{\frac{n}{m}}\right)^2.$$

## IX.

1. Shew that the directrices of all parabolas touching a curve of finite curvature at any given point, and having the same curvature at that point as the curve, pass through a fixed point.

2. Determine the parabola in magnitude and position for a point in the circle, when the subtenses are inclined at $45^0$ to the tangent.

3. Find the focus of the parabola of curvature which

touches a cycloid at its vertex, and the locus of the foci of all parabolas which have the same tangent and curvature at that point.

4. If $AEB$ be the chord, $AD$ the tangent, and $BD$ the subtense, for an arc $ACB$ of finite curvature at $A$, find the limit of the area $ACBE$ : area $ACBD$ as $B$ approaches $A$.

5. If from a point $A$, equal arcs $AB$, $AC$ be measured and $BC$ be joined, shew that $BC$ is ultimately parallel to the tangent at $A$.

6. Apply the property that every curve of finite curvature is ultimately a parabola, to shew that two tangents drawn from a point to the extremities of a small arc are ultimately equal, when the arc is diminished indefinitely.

7. In a segment of an arc of finite curvature a pentagon is inscribed one side of which is the chord of the arc, and the remaining sides equal. Shew that the limiting ratio of the areas of the pentagon and segment, when the chord moves up towards the tangent at one extremity is 15 : 16.

8. $APQ$ is a curve of continued and finite curvature, $P$ and $Q$ are two points in it, whose abscissæ along the normal at $A$ are always in the ratio $m : 1$, and from $B$, $C$ two points in the normal, straight lines $BPb$, $CPc$, $BQb'$, $CQc'$ are drawn to meet the tangent at $A$. Shew that when $P$ and $Q$ move up to $A$, the areas of the triangles $bPc$, $b'Qc'$ are ultimately in the ratio $m^{\frac{5}{2}} : 1$.

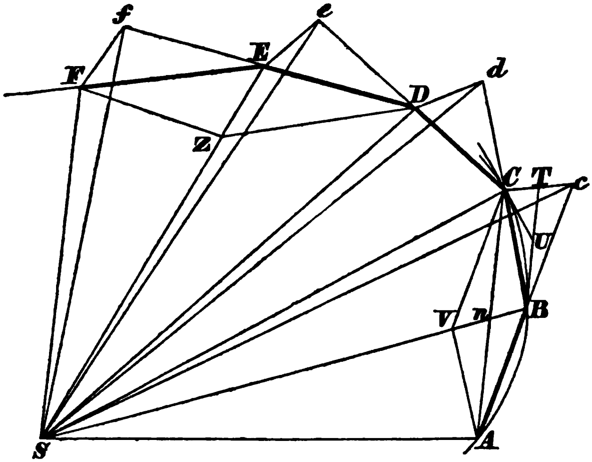
## SECTION II.

### *Centripetal Forces.*

Prop. I. Theorem I.

*When a body revolves in an orbit, subject to the action of forces tending to a fixed point, the areas, which it describes by radii drawn to the fixed center of force, are in one fixed plane, and are proportional to the time of describing them.*

Let the time be divided into equal parts, and in the first interval let the body describe the straight line *AB* with




uniform velocity, being acted on by no force. In the second interval it would, if no force acted, proceed to *c* in *AB* produced, describing *Bc* equal to *AB*: so that the equal areas *ASB*, *BSc* described by radii *AS*, *BS*, *cS* drawn to the center *S*, would be completed in equal intervals.

But, when the body arrives at *B*, let a centripetal force tending to *S* act upon it by a single instantaneous impulse, and cause the body to deviate from the direction *Bc* and to proceed in the direction *BC*.

Let *cC* be drawn parallel to *BS*, meeting *BC*, then, at the end of the second interval, the body will be found at *C*, in the same plane with the triangle *ASB*, in which *Bc* and *cC* are drawn. Join *SC*; and the triangle *SBC*, between parallels *SB*, *Cc* will be equal to the triangle *SBc*, and therefore also to the triangle *SAB*.

In like manner, if the centripetal force act upon the body successively at *C*, *D*, *E*, &c. causing the body to describe in the successive intervals of time the straight lines *CD*, *DE*, *EF*, &c. these will all lie in the same plane; and the triangle *SCD* will be equal to the triangle *SBC*, and *SDE* to *SCD*, and *SEF* to *SDE*.

Therefore equal areas are described in the same fixed plane in equal intervals; and, componendo, the sums of any number of areas *SADS*, *SAFS*, are to each other as the times of describing them.

Let now the number of these triangles be increased, and their breadth diminished indefinitely; then their perimeter *ADF* will be ultimately a curve line; and the instantaneous forces will become ultimately a centripetal force, by the action of which the body is continually deflected from the tangent to this curve, and which will act continuously; and the areas *SADS*, *SAFS*, being always proportional to the times of describing them, will be so in this case. Q.E.D.

Cor. 1. The velocity of a body attracted towards a fixed center in a non-resisting medium, is reciprocally proportional to the perpendicular dropped from that center upon the tangent to the orbit.

For the velocity at the points *A*, *B*, *C*, *D*, *E* are as the bases *AB*, *BC*, *CD*, *DE*, *EF* of equal triangles, and since the triangles are equal these bases are reciprocally proportional to the perpendiculars from *S* let fall upon them. [And the same is true in the limit, in which case the bases are in the direction of tangents to the curvilinear limit, therefore the velocity, &c.]

COR. 2. If on chords *AB*, *BC* of two arcs described in equal successive times in a non-resisting medium by the same body the parallelogram *ABCV* be completed, and the diagonal *BV* of this parallelogram be produced in both directions in that position which it assumes ultimately when those arcs are diminished indefinitely, it will pass through the center of force.

COR. 3. If on *AB*, *BC* and on *DE*, *EF* chords of arcs described in a non-resisting medium in equal times, the parallelograms *ABCV*, *DEFZ* be completed; the forces at *B* and *E* are to one another in the ultimate ratio of the diagonals *BV*, *EZ*, when the arcs are indefinitely diminished.

For the motions of the body represented by *BC*, *EF* in the polygon, are compounded of the motions represented by *Bc*, *BV* and *Ef*, *EZ*; and those represented by *BV*, *EZ* which are equal to *Cc*, *Ff* in the demonstration of the proposition were generated by the impulses of the centripetal force at *B* and *E* and are thus proportional to those impulses. [And the same is true in the limit, in which case the ultimate ratio of the impulses at any two points is the ratio of the continuous forces at those points.]

COR. 4. The forces by which any bodies moving in non-resisting media are deflected from rectilinear motion into curved orbits, are to one another as those sagittæ

of arcs described in equal times, which converge to the center of force and bisect the chords, when those arcs are indefinitely diminished.

For the diagonals of the parallelograms *ABCV*, *DEFZ* bisect each other, and these sagittæ are halves of the diagonals *BV*, *EZ* when the arcs are indefinitely diminished. [And the same is true whether *ABCV* and *DEFZ* be parts of the same or of different orbits if the arcs be described in equal times.]

Cor. 5. And therefore the same forces are to the force of gravity as those sagittæ are to vertical sagittæ of the parabolic arcs which projectiles describe in the same time.

Cor. 6. All the same conclusions obtain, by the Second Law of Motion, when the planes, in which the bodies move together with the centers of force which are situated in those planes, are not at rest, but are moving uniformly and parallel to themselves.

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The statement of the proposition in the original Latin is,

"Areas, quas corpora in gyros acta radiis ad immobile centrum virium ductis describunt, et in planis immobilibus consistere, et esse temporibus proportionales."

---

90. It should be carefully observed, that, before proceeding to the limit, it is proved that *any* polygonal areas *SADS*, *SAFS*, are proportional to the times of description of their perimeters; so that ultimately these areas become *finite* curvilinear areas, described in finite times.

91. In proceeding to the ultimate state of the hypothesis, it is concluded readily from Lemmas II, and III, that the curvilinear areas are the limits of the polygons; but a greater

difficulty arises in the transition from the *discontinuous* motion under the action of instantaneous impulsive forces to the *continuous* motion under the action of a continuous force tending to $S$. For, in the curvilinear path of the body which is the limit of the perimeter of the polygon, the direction of the motion at the angular points of the polygon is different, and also the deflection from the direction of motion is twice as great in the polygon as it is in the curve. Although we may assume that the curvilinear limit of the perimeter of the polygon may be described under the action of some force, is that force the same which is the limit of the series of impulses?

The centripetal force supposed to act with a simple instantaneous impulse, "impulso unico et magno," is supposed to generate a finite velocity at once, which effect a finite force cannot produce.

If instead of this imaginary impulse we suppose the force finite, but very great, and acting for a very short time, the effect upon the figure would be to round off the angular points of the polygon.

The transition from the impulses to the continuous force, in the ultimate form of the hypothesis, must be considered as axiomatic, like the ultimate equality of the ratio of the finite arc to the perimeter of the inscribed polygon.

92. We can, however, shew that if the curvilinear limit of the polygon be described under the action of some continuous force tending to $S$, the effect of this force estimated by the quantity of motion generated in the interval between the impulses, is ultimately the same as that generated by the impulse.

Consider first the geometrical properties of the limit of the polygonal perimeter.

Let $BT$, $CU$ be tangents at $B$, $C$, to the curvilinear limit, and let $Cc$ intersect $BT$ in $T$, (fig. page 119.)

Now, since $Cc$ ultimately vanishes compared with $Bc$, $BC$

and $Bc$ or $AB$ are ultimately in a ratio of equality, and $Cc$ is ultimately bisected by $BT$ (Art. 38); also, $CU = BU = UT$, ultimately. (Art. 82).

Consider next, the effects produced by the different kinds of force which act in the two cases.

In the polygonal path, the impulsive force at $B$ generates a velocity with which the body describes $Cc$ in the time $t$, in which $AB$ or $BC$ is described, the measure of the effect of the force is therefore the velocity $\frac{Cc}{t}$.

In the curvilinear path the deflection from the direction $BT$ at $B$ in the same time $t$ is $TC$ by the continuous action of finite forces, if we suppose the force ultimately uniform in magnitude and direction the measure of the accelerating effect of the force is $\frac{2TC}{t^2}$, and the velocity generated in that time is

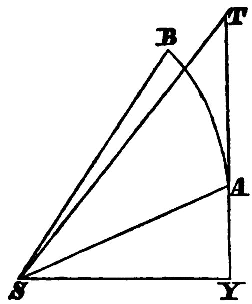
$$\frac{2TC}{t^2} \cdot t = \frac{Cc}{t}.$$

Hence the effects of the finite and impulsive forces measured by the quantity of motion produced are the same.

93. To shew that a continuous force, which generates the same quantity of motion as the impulse at $B$ in the time from $B$ to $C$, would cause the body on arriving at $C$ to move in the direction of the tangent to the curvilinear limit of the perimeter.

The velocity due to the action of the finite force at the end of time $t$ being ultimately $\frac{2TC}{t}$ in the direction $TC$ and that in the direction $BT$ being $\frac{BT}{t} = \frac{2TU}{t}$; therefore $TC$, $UT$ represent the velocities in those directions; therefore $UC$ is the direction of motion at $C$; or the body moves in the direction of the tangent at $C$.

94. Cor. 1. The corollary may be proved directly from the proposition, for the proportionality of the areas to the times of describing them is true if the force suddenly cease to act, in which case the body proceeds in the direction of the tangent.




Let $V$ be the velocity at the point $A$, $ASB$ the curvilinear areas described in any time $T$, $AT = V.T$ the space described if the force cease to act.

Join $ST$ and draw $SY$ perpendicular to $AT$,

then area $ASB$ = triangle $SAT$

$$= \tfrac{1}{2} V.T \times SY,$$

and area $ASB \propto T$;

$$\therefore\ V \propto \frac{1}{SY}.$$

Also, if $h$ be twice the area described in the unit of time employed in estimating the accelerating effect of the force tending to $S$, and the velocity $V$ of the body,

$$2 \,.\, \text{area } SAB = hT;$$

$$\therefore\ h = V.SY.$$

By the use of this area the proportions employed by Newton may be converted into equations, for the convenience of calculation.

If bodies move in curves for which the areas, described in the same time, are not equal

$$V \propto \frac{h}{SY}.$$

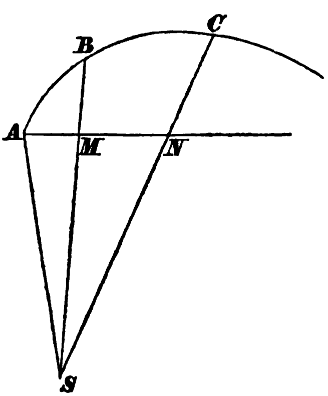
95. Cor. 5. The object of this corollary is to determine the numerical measure of the central force which governs the motion of a body, when the circumstances of the motion are known: for it supplies us with the ratio of this force to the force

of gravity at any place, the measure of which is determined by experiment.

*Application of the Proposition.*

96. Prop. When the force instead of tending to a fixed point, acts in parallel lines, the property of the motion enunciated in the proposition may be replaced by the property that the resolved part of the space described perpendicular to the direction of the force is proportional to the times.

This is immediately deducible from the second law of motion, since there is no force in the direction perpendicular to that of the forces, and the velocity in that direction is uniform.




That this is the result of the properties in the proposition may be shewn by removing the center of force to an infinite distance.

If $S$ be the center of force, $AMN$ perpendicular to $SB$, the area $ABCS$ is proportional to the time of describing $AC$, and the area $AMNS$ and $ABCS$ are ultimately equal when $S$ is removed to an infinite distance in $BMS$, $\therefore$ the triangle $ASN$ is proportional to the time and the base $AN$ is so also, which since $CN$ is ultimately perpendicular to $AN$ is the proposition required.

97. Prop. If a body describe a curvilinear orbit about a force tending constantly to a fixed point, the area described in a given time will be unaltered, if the force be suddenly increased or diminished, or if the body be acted on at any moment by an impulsive force tending to that point.

For, if in the polygon the impulse at any point $B$ be increased or diminished by any force tending to or from $S$, the only effect is to remove the vertex $C$ of the triangle $SBC$ to some other point in the line

$cC$ parallel to $BS$, and the area will be unaltered, and the argument which establishes the equality of polygonal areas in a given time proceeds as before.

In the limit the curvilinear areas in a given time are unaltered.

If at $B$ the new force introduced be impulsive, the angle $ABC$ remains less than two right angles when we proceed to the limit, and the parts of the curve cut one another at a finite angle.

Hence, in any calculation the value of $h$ (Art. 94), is the same before and after the change of the force.

*Illustrations.*

1. If a body describe an ellipse under the action of a force tending to one of the foci, the square of the velocity varies inversely as the distance from that focus, and directly as the distance from the other.

The square of the velocity $\propto \dfrac{1}{SY^2} \propto HZ^2$,

$$\text{and } HZ : SY :: HP : SP;$$

$$\therefore HZ.SY : SY^2 :: HP : SP,$$

$$\text{and } HZ.SY = BC^2; \quad \therefore SY^2 \propto \frac{SP}{HP},$$

$$\text{and the (velocity)}^2 \propto \frac{HP}{SP}.$$

2. The velocity is greatest when the body is at the extremity of the major axis which is nearest to the focus to which the force tends, and least at the other extremity.

$SY$ being the greatest in the first and least in the second position,

3. The velocity is a geometric mean between the greatest and least velocities at the extremities of the minor axis.

For at this point $HZ = BC$, and at the extremities of the major axis the values of $HZ$ are $SA$ and $Sa$,

$$\text{and } BC^2 = SA.Sa.$$

4. In the equiangular spiral described under the action of a force tending to the focus, the velocity $\propto \dfrac{1}{SP}$.

For, $SY \propto SP$.

5. If the force tends to the center of the elliptic orbit described by a body, the time between the extremities of conjugate diameters is constant.

For the area $PCD$ is constant.

## X.

1. If different bodies be projected with the same velocity from a given point, all being attracted by forces tending to one fixed point, shew that the areas described by the lines drawn from the fixed points to the bodies are proportional to the sines of the angles of projection.

2. When a body describes a curvilinear orbit under the action of a force tending to a fixed point, will the direction of motion or the curvature of the orbit at any point be changed if the force at the point receive a finite change?

3. If an ellipse be described under the action of a force tending to the center, find at what points the velocity is the greatest and least.

4. Shew that in the case of 3, the velocity varies directly as the diameter conjugate to that which passes through the body.

5. Shew that the sum of the squares of the velocities at the extremities of conjugate diameters is constant.

6. If the velocity at any point can be equal to the difference of the greatest and least velocities the major axis must be less than double of the minor.

7. If a body describe an ellipse under the action of a central force tending to one of the foci, shew that the sum of the velocities at the extremities of the two latera recta varies in-

versely as the diameters parallel to the direction of motion at those points.

8. In a parabolic orbit described round a force tending to the focus, the velocity varies inversely as the normal at any point.

9. In the orbit of 8, shew that the sum of the squares of the velocities at the extremities of a focal chord is constant.

10. In an ellipse described round a force tending to the focus, compare the intervals of time between the extremities of the same latus rectum when $AC = 2CS$.

11. In the case of 10, the time of moving from the nearest focal distance to the extremity of the minor axis is $m$ times that from the extremity of the minor axis to the greatest focal distance; find the eccentricity, and shew that, if there be a small error in $m$, the corresponding error in the eccentricity varies as $\dfrac{1}{(1+m)^2}$.

12. The velocity in a cardioid described about a force tending to the pole varies in the inverse sesquiplicate ratio of the distance.

13. The velocity in the Lemniscate varies inversely as the cube of the central distance, when a particle moves in the curve round a force tending to the center.

14. A body describes a parabola about the focus; if the segments $PS$, $Sp$ of the focal chord $PSp$ be in the ratio $n : 1$, prove that the time of describing $pA$ : time of describing $AP$

$$:: 3n + 1 : n^2(n + 3).$$

15. In motion in an ellipse round a force tending to the center, shew that the velocity at any point perpendicular to either focal distance is constant; and the sum of the squares of the velocities at the extremities of any pair of semi-conjugate diameters resolved in any given direction is constant.

16. In the motion supposed in 15, having given any point $P$ in the ellipse, determine geometrically the points $p_1, p_2, p_3$, &c... so that the time in $Pp_1$, $p_1p_2$, $p_2p_3$, ... are each equal to $\frac{1}{n}$-th of the periodic time.

Also shew that if the times in $AP_1$, $P_1P_2$, $P_2P_3$, $P_3R$ be equal, and $v$, $v_1$, $v_2$, $v_3$, $v'$, be the velocities at $A$, $P_1$, $P_2$, $P_3$, $B$ respectively,

$$2(v_1^2 + v_2^2 + v_3^2) = 3(v^2 + v'^2).$$

17. In the ellipse described about the focus $S$, $ASHA'$ being the major axis, time in $AB$ : time in $BA'$ :: $\pi + 2e : \pi - 2e$.

18. Prove that in an equiangular spiral described by a body about a force in the focus the time in any arc varies as the difference of the squares of the focal distance of the extremities.

## Prop. II. Theorem II.

*Every body, which moves in any curve line described in a plane, and describes areas proportional to the times of describing them about a point either fixed or moving uniformly in a straight line, by radii drawn to that point, is acted on by a centripetal force tending to the same point.*

Case 1. Let the time be divided into equal intervals, and, in the first interval, let the body describe $AB$ with uniform velocity, being acted on by no force; in the second interval it would, if no force acted, proceed to $c$ in $AB$ produced, describing $Bc$ equal to $AB$; and the triangles $ASB$, $BSc$ would be equal. But, when the body arrives at $B$, let a force acting upon it by a single impulse, cause the body to describe $BC$ in the second interval of time so that the triangle $BSC$ is equal to the triangle

*ASB*, and therefore also to the triangle *BSc*; ∴ *BSC* and *BSc* are between the same parallels, ∴ *BS* is parallel to *cC*, ∴ *BS* was the direction of the impulse at *B*.

Similarly, if at *C*, *D*, ... the body be acted on by impulses causing it to move in the sides *CD*, *DE*, ... of a polygon, in the successive intervals, making the triangles *CSD*, *DSE*, ... equal to *ASB* and *BSC*, the impulses can be shewn to have been in the directions *CS*, *DS*... Hence, if *any* polygonal areas be described proportional to the times of describing them, the impulses all tend to *S*.

The same is true if the number of intervals be increased and their length diminished indefinitely, in which case the series of impulses approximates to a continuous force tending to *S*, and the polygons to curvilinear areas, as their limits. Hence the proposition is true for a *fixed* center.

Case 2. The proposition will also be true, if *S* be a point which moves uniformly in a straight line, for, by the second law of motion, the relative motion will be the same, whether we suppose the plane to be at rest, or that it moves together with the body which revolves and the point *S*, uniformly in one direction.

Cor. 1. In non-resisting media, if the areas are not proportional to the times, the forces do not tend to the point to which the radii are drawn, but deviate *in consequentiâ*, i. e. in that direction towards which the motion takes place, if the description of areas is accelerated; but if it be retarded the deviation is *in antecedentiâ*.

Cor. 2. And also in resisting media, if the description of areas is accelerated, the directions of the forces deviate from the point to which the radii are drawn in that direction towards which the motion takes place.

## SCHOLIUM.

A body may be acted on by a centripetal force compounded of several forces. In this case, the meaning of the proposition is, that that force, which is the resultant of all, tends to $S$. Moreover, if any force act continually in a line perpendicular to the plane of the areas described, this force will cause the body to deviate from the plane of its motion, but will neither increase nor diminish the amount of area described, and therefore must be neglected in the composition of the forces.

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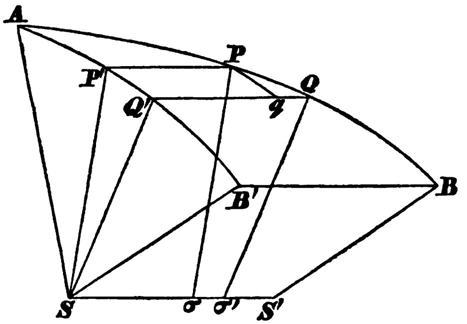
98. The description of an area round a point in motion may be explained as follows.




Let $SS'$ be the line in which $S$ moves uniformly and let the body move from $A$ to $B$ in the same time as $S$ moves from $S$ to $S'$, $P$, $\sigma$ simultaneous positions of the body and $S$.

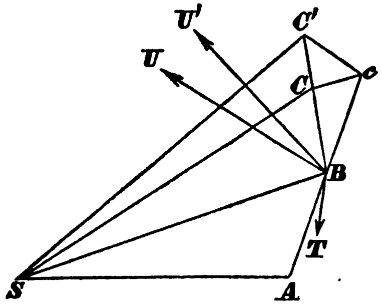
If $PP'$ be drawn equal and parallel to $\sigma S$, and the same construction be made for every point in the path of the body, the curve $AP'B'$, which is the locus of $P'$, is the orbit which the body would appear to describe to an observer at $S$, who refers all the motion to the body.

This is clear, since $SP'$ is equal and parallel to $\sigma P$, and the

distance of the body and the direction in which it is seen is the same in the two cases.

If $Q$ $Q'$ be corresponding points near $P$ and $P'$, and the force at $\sigma$ be supposed to act impulsively, the relative motion round $\sigma$ will be unaltered if we apply equal velocities in the same direction to $P$ and $\sigma$, so that $\sigma$ is reduced to rest, $\sigma\sigma'$ and $PQ$ are described in the same time, take therefore $Qq$ equal and parallel to $\sigma'\sigma$, the resulting motion of $P$ is then $Pq$ by the second law of motion, which is equal and parallel to $P'Q'$; therefore $P'Q'$ represents the relative motion of the body round $S$ at rest.

99. Cor. 1. Reverting to the polygonal area, if the tri-




angle $SBC'$ be greater than the triangle $SAB$, the impulse at $B$ is not in the direction $BS$ but $BU$, parallel to $cC'$, or if the areas are not proportional to the times but are in an increasing ratio, the direction of the force deviates towards the direction in which the description of areas is accelerated: and *vice versâ* when the description is retarded.

100. Cor. 2. The effect of a resisting medium is to retard the motion, or, supposing it to act by impulses, we must conceive an impulse at $B$ in the case of the polygon in the direction $BA$; if therefore the description of areas be accelerated, the impulse applied at $B$ in the direction $BU'$ acts still further

*in consequentiâ*, so that this impulse with the impulse corresponding to the resistance of the medium may produce a resultant impulse in the direction of $BU$.

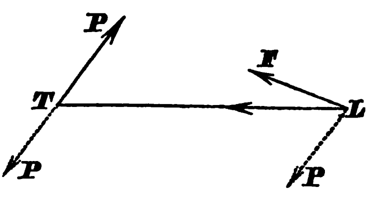
The effect of the resistance alone is to retard the description of areas.

If the force act *in consequentiâ*, the resistance of this force and the resistance of the medium may act in the direction $BS$, and the proportionality of the areas to the time be preserved.

### Prop. III. Theorem III.

*Every body, which describes areas proportional to the times of describing them by radii drawn to the center of another body which is moving in any manner whatever, is acted on by a force compounded of a centripetal force tending to that other body, and of the whole accelerating force which acts upon that other body.*

Let the first body be $L$, the second $T$, $T$ moves under the action of some force $P$, $L$ under the action of another force $F$. At every instant apply to both bodies the force $P$ in the contrary direction to that in which it acts, as represented by the dotted arrows.




$L$ will continue to describe about $T$, as before, areas proportional to the times of describing them, and since there is now no force acting on $T$, $T$ is at rest or moves uniformly in a straight line.

Therefore, (by Theor. 2) the resultant of the force $F$ and the force $P$ applied to $L$ tends to $T$.

Hence $F$ is compounded of a centripetal force tending to $T$, and of a force equal to that which acts on $L$. Q.E.D.

COR. 1. Hence if a body $L$ describes areas proportional to the times of describing them by radii drawn to another body $T$; and from the whole force, which acts upon $L$, whether a single force or compounded of several forces, be taken away the whole accelerating force which acts upon the other body $T$; the whole remaining force, which acts upon $L$, will tend to the other body $T$ as a center.

COR. 2. And, if these areas are very nearly proportional to the times of describing them, the remaining force will tend to the other body very nearly.

COR. 3. And conversely, if the remaining force tends very nearly to the other body $T$, the areas will be very nearly proportional to the times.

COR. 4. If the body $L$ describes areas which are very far from being proportional to the times of describing them, by radii drawn to another body $T$; and that other body $T$ is at rest, or moves uniformly in a straight line: then, either there is no centripetal force tending to that other body $T$, or such centripetal force is compounded with the action of other very powerful forces, and the whole force, compounded of all the forces, if there be many, is directed towards some other center fixed or moving.

The same holds, when the other body moves in any manner whatever; if the centripetal force spoken of be understood to be that which remains after taking away the whole force acting upon the other body $T$.

SCHOLIUM.

Since the equable description of areas is a guide to the center to which that force tends, by which a body is principally acted on, and by which it is deflected from rectilinear motion, and retained in its orbit; we may in

what follows employ the equable description of areas as a guide to the center, about which all curvilinear motion in free space takes place.

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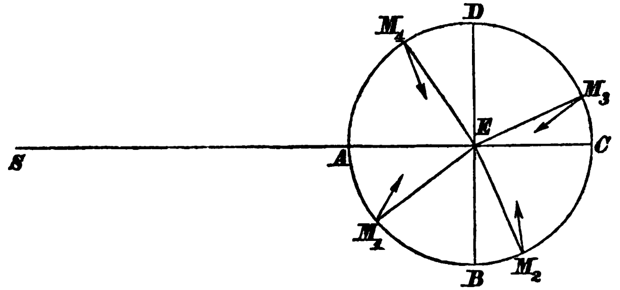
101. As an illustration of the last propositions and their corollaries, we may examine some of the circumstances of the motion of the Moon, Earth, and Sun.

Suppose the Moon's orbit relative to the Earth to be nearly circular, and let $ABCD$ be this orbit, $E$ the Earth.




1. The areas described by the radii drawn from the Moon to the Earth are nearly proportional to the times of describing; hence the resultant force on the Moon tends nearly to $E$.

2. If $ES$ the line joining the centers of the Earth and Sun meets the Moon's relative orbit in $A$, $C$, and $DEB$ be perpendicular to $ES$, the description of areas is accelerated as the Moon moves from $D$ to $A$ and from $B$ to $C$, and retarded from $A$ to $B$ and from $C$ to $D$; hence the direction of the resultant force on the Moon in the positions $M_1$, $M_2$, $M_3$, $M_4$, is in the directions of the arrows slightly inclined to the radii drawn to $E$.

From these observed facts, we see that when the force, under the action of which $E$ moves, is applied to the Moon in the contrary direction, the remaining force tends in the directions of the arrows.

By the supposition that the Earth and Moon are acted on by forces tending to the Sun, whose distance compared with

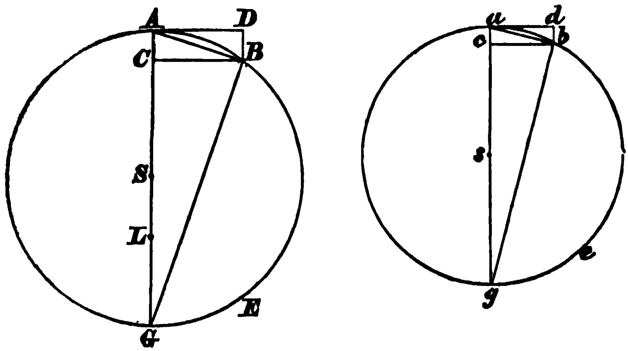
*EM* is very great, and that the differences of the forces on these bodies are not very great, the circumstances of the description of areas in the motion of the Moon are accounted for.

## PROP. IV. THEOREM IV.

*The centripetal forces of bodies, which describe different circles with uniform velocity, tend to the centers of the circles, and are to each other as the squares of areas described in the same time, divided by the radii of the circles.*

The bodies move uniformly, therefore the arcs described are proportional to the times of describing them; and the sectors of circles are proportional to the arcs on which they stand, therefore the areas described by radii drawn to the centers are proportional to the times of describing them; hence, by Prop. II, the forces tend to the centers of the circles.

Again, let $AB$, $ab$ be small arcs described in equal times,




$AD$, $ad$ tangents at $A$, $a$, $ACSG$, $acsg$ diameters through $A$, $a$. Join $AB$, $ab$, and draw $BC$, $bc$ perpendicular to $AG$, $ag$.

By similar triangles, $AC : AB :: AB : AG$,

$$\therefore AC \cdot AG = (\text{chord } AB)^2;$$

$$\therefore AC : ac :: \frac{(\text{chord } AB)^2}{AG} : \frac{(\text{chord } ab)^2}{ag}.$$

But ultimately when the arcs $AB$, $ab$ are indefinitely diminished, since $AC$, $ac$ are sagittæ of the double of arcs $AB$, $ab$, and are therefore, by Prop. I. Cor. 4, ultimately as the forces at $A$ and $a$;

$\therefore$ ultimately, force at $A$ : force at $a$

$$:: \frac{(\text{chord } AB)^2}{AG} : \frac{(\text{chord } ab)^2}{ag}$$

$$:: \frac{(\text{arc } AB)^2}{AG} : \frac{(\text{arc } ab)^2}{ag}, \text{ by Lemma VII.}$$

Take $AE$ $ae$ two arcs described in *any* equal finite times, $\therefore AE : ae :: AB : ab$ since the bodies move uniformly, and this is also true in the limit;

$$\therefore \text{force at } A : \text{force at } a :: \frac{AE^2}{AS} : \frac{ae^2}{as}.$$

Q. E. D.

COR. 1. Since these arcs are proportional to the velocities of the bodies, the centripetal forces will be in the ratio compounded of the duplicate ratio of the velocities directly, and the simple ratio of the radii inversely.

That is, if $V$, $v$ be the velocity, $R$, $r$ the radii in two circles, $F$, $f$ the centripetal forces,

$$AE : ae :: V : v;$$

$$\therefore F : f :: \frac{V^2}{R} : \frac{v^2}{r}.$$

COR. 2. And since the circumferences of the circles are described in their periodic times, the velocities are in the ratio compounded of the ratio of the radii directly and the ratio of the periodic times inversely; hence the

centripetal forces are in the ratio compounded of the ratio of the radii directly, and of the ratio of the periodic times inversely.

If $P, p$ be the periodic times in the two circles respectively,

$$V : v :: \frac{2\pi R}{P} : \frac{2\pi r}{P} :: \frac{R}{P} : \frac{r}{p};$$

$$\therefore\ F : f :: \frac{R}{P^2} : \frac{r}{p^2}.$$

Cor. 3. Hence, if the periodic times be equal, and therefore the velocities proportional to the radii, the centripetal forces will be as the radii; and conversely.

$$\text{If } P = p,\ V : v :: R : r;$$

$$\therefore\ F : f :: \frac{V^2}{R} : \frac{v^2}{r} :: R : r.$$

Cor. 4. Also if the periodic times are in the subduplicate ratio of the radii, the centripetal forces are equal.

$$\text{For, } P^2 : p^2 :: R : r.$$

Cor. 5. If the periodic times are as the radii, and therefore the velocities equal, the centripetal forces are reciprocally as the radii, and conversely.

Cor. 6. If the periodic times are in the sesquiplicate ratio of the radii, and therefore the velocities reciprocally in the subduplicate ratio of the radii, the centripetal forces are reciprocally as the squares of the radii; and conversely.

That is, if $P^2 : p^2 :: R^3 : r^3$;

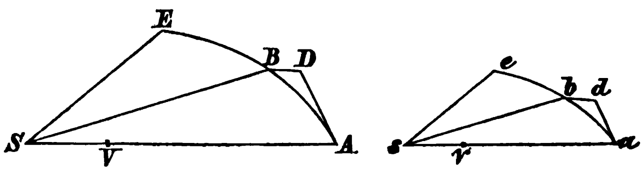
$$\therefore\ V^2 : v^2 :: \frac{R^2}{P^2} : \frac{r^2}{p^2} :: r : R;$$

$$\therefore\ F : f :: \frac{r}{R} : \frac{R}{r} :: r^2 : R^2.$$

Cor. 7. And, generally, if the periodic times vary as any power $R^n$ of the radius $R$, and, therefore, the velocity vary inversely as the power $R^{n-1}$; the centripetal force will vary inversely as $R^{2n-1}$; and conversely.

Cor. 8. All the same proportions can be proved concerning the times, velocities, and forces, by which bodies describe similar parts of any figures whatever which are similar and have centers of force similarly situated, if the demonstrations be applied to those cases, *uniform description of areas* being substituted for *uniform velocity*, and *distances of the bodies from the centers of force* for *radii of the circles*.

Let $AE$, $ae$ be similar arcs of similar curves described by bodies about forces tending to similarly situated points $S$, $s$; $AB$, $ab$ small arcs described in *equal* times $T$, $BD$,




$bd$ subtenses parallel to $SA$, $sa$, $AV$, $av$ chords of curvature at $A$, $a$, so that by similar figures we have

$$AV : av :: AS : as.$$

Then force at $A$ : force at $a$ :: $DB : db$, ultimately,

$$:: \frac{AB^2}{AV} : \frac{ab^2}{av} :: \frac{AB^2}{SA} : \frac{ab^2}{sa}, \text{ ultimately};$$

and if $V$, $v$ be the velocities at $A$, $a$, since $AB$, $ab$ are described in equal times,

$$AB : ab :: V : v, \text{ ultimately};$$

$$\therefore \text{ force at } A : \text{force at } a :: \frac{V^2}{SA} : \frac{v^2}{sa},$$

corresponding to Cor. 1.

Again, if $AB$, $ab$ be small *similar* arcs described in times $T$, $t$, instead of being arcs described in equal times, and $P$, $p$ be the times of describing similar finite arcs $AE$, $ae$,

$$T : P :: \text{area } ASB : \text{area } ASE$$
$$:: \text{area } asb : \text{area } ase :: t : p;$$
$$\therefore T : t :: P : p;$$

and this being true always is true when $AB$, $ab$ are indefinitely diminished.

$$\text{Hence, } F : f :: \frac{BD}{T^2} : \frac{bd}{t^2}, \text{ ultimately,}$$
$$:: \frac{SA}{P^2} : \frac{sa}{p^2},$$

corresponding to Cor. 2.

Cor. 9. It follows also from the same proposition, that the arc, which a body moving with uniform velocity in a circle under the action of a given centripetal force describes in any time, is a mean proportional between the diameter of the circle, and the space through which the body would fall from rest under the action of the same force and in the same time.

For let $AL$ be the space described from rest in the same time as the arc $AE$, then since if $BD$ be perpendicular to the tangent at $A$, $BD$ is ultimately the space described by the body by the action of the force in the time in which the body describes the arc $AB$, and the times are proportional to the arcs:

$$\therefore AL : BD :: AE^2 : AB^2;$$
$$\therefore AL.AG : BD.AG :: AE^2 : AB^2;$$
$$\therefore \text{since } BD.AG = (\text{chord } AB)^2 = (\text{arc } AB)^2, \text{ ultimately};$$
$$AL.AG = AE^2, \text{ or } AL : AE :: AE : AG.$$

Q. E. D.

SCHOLIUM.

The case of the sixth Corollary holds for the heavenly bodies, and on that account the motion of bodies acted upon by a centripetal force, which decreases in the duplicate ratio of the distance from the center of force, is treated of more fully in the following section.

Moreover, by the aid of the preceding proposition and its corollaries, the proportion of a centripetal force to any known force, such as gravity, can be obtained. For, if a body revolve in a circle concentric with the earth by the action of its own gravity, this gravity is its centripetal force.

But from the falling of heavy bodies by Cor. 9, both the time of one revolution and the arcs described in any given time are determined.

And by propositions of this kind Huygens in his excellent tract, *De Horologio Oscillatorio*, compared the force of gravity with the centrifugal force of revolving bodies.

The preceding results may be proved in this manner. In any circle let a regular polygon be supposed to be described of any number of sides. And if a body moving with a given velocity along the sides of the polygon be reflected by the circle at each of its angular points, the force with which it impinges on the circle at each of the reflections, will be proportional to the velocity; and therefore the sum of the forces, in a given time, will vary as the velocity and the number of the reflections conjointly. But if the number of sides of the polygon

be given, the velocity varies as the space described in a given time, and the number of reflections in a given time varies, in different circles, inversely as the radii of the circles, and, in the same circle, directly as the velocity. Hence, the sum of the forces exerted in a given time varies as the space described in that time increased or diminished in the ratio of that space to the radius of the circle; that is, as the square of that space divided by the radius, and therefore, if the number of sides be diminished indefinitely so that the polygon coincides with the circle, the sum of the forces varies as the squares of the arc described in the given time divided by the radius.

This is the *centrifugal force* by which the body presses against the circle, and to this the opposite force is equal, by which the circle continually repels the body towards the center.

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*Symbolical representation of Areas, Lines, &c.*

102. In the statement of the proposition the words "arcuum quadrata applicata ad radios" in the text of Newton, is rendered the squares of arcs divided by the radii. Such expressions as $\frac{AB^2}{AG}$ may be regarded as representations of lines, (*e.g.* this expression denotes $AC$,) whose lengths are determined by such constructions as the following:

To $AG$ apply a rectangle whose area is that of the square on $AB$, and let $AC$ be the side adjacent to $AG$; $AC$ is thus obtained by applying the square on $AB$ to $AG$. The propriety of the symbol $\frac{AB^2}{AG}$ employed to represent a line $AC$, assumed from algebra, is obvious, since the number of units of area

in the square on $AB$ and in the rectangle whose sides are $AG$, $AC$ are the same, hence if $m$, $n$, $r$ be the number of units of length in these lines $m^2 = n \times r$, and $r = \frac{m^2}{n}$.

103. If symbols of this kind, viz. $\frac{AB^2}{AG}$, be used in the same manner as a fraction, we may either treat them numerically, considering $AB^2$ to represent the number of units of area contained in the square on $AB$, and $AG$ as the number of units of length in $AG$, and thus apply the rules of Arithmetical Algebra; or, we may look upon $AB^2$ as the absolute representation of an area, and $AG$ as that of a line, in which case $\frac{AB^2}{AG}$ has no meaning except by interpretation. In this interpretation we are guided by the principles upon which Symbolical Algebra is applied to any science, the laws of operation by symbols being the same in Arithmetical and Symbolical Algebra, and the symbols being interpreted so that these laws are not contradicted. Thus, in the application to Geometry, the symbol $A$ being supposed to denote an area equal to that of a rectangle whose sides are represented by $a$ and $b$, we assume that $A = a \, . \, b$; hence, the laws being the same in Arithmetical and Symbolical Algebra, $\frac{A}{a} = b$; whence the interpretation, if a rectangle be applied to $a$, whose area is $A$, $\frac{A}{a}$ denotes the other side of the rectangle.

*Notes.*

104. Cor. 1. This corollary asserts that the centripetal force at any point in the circle varies as $\frac{V^2}{R}$, but it is further true that, if $F$ be the measure of the accelerating effect of the force, $F \, . \, R = V^2$, or that, if a body move from rest under the

action of a constant force whose accelerating effect is the same as that of the centripetal force, the velocity generated in passing through a space equal to half the radius of the circle is equal to the velocity with which the circle is uniformly described.

For, referring to the figure used in the proposition, $DB$ is the space through which the body is deflected from the tangent in the time $T$ in which $AB$ is described with the velocity $V$; and this is ultimately the space through which the body would be drawn from rest in the time $T$ by the force at $A$ continued constant in direction and magnitude;

$$\therefore\ DB = \frac{FT^2}{2}\text{, ultimately,}$$

$$\text{and } AB = V.T;$$

$$\therefore\ DB.AG = V^2.T^2;$$

$$\therefore\ V^2 = F\frac{AG}{2} = F.R.$$



105. Scholium. In uniform circular motion the centripetal force is employed in counteracting the tendency of the body to move in a straight line, which it would do, according to the first law of motion, with the uniform velocity which it has at any point of the circle, if the centripetal force were suddenly to cease to act. This tendency to recede is improperly called a centrifugal *force;* for the effect of a force being to accelerate or retard the motion of a body, or to alter its direction, if the tendency could properly be termed a force and the centripetal force which counteracts it were removed, it would accelerate or retard the motion of the body, or alter its direction, which it does not.

106. In this case of circular motion the force is exerted not in accelerating or retarding the motion, but in changing its direction.

Thus, referring to the figure of Prop. I., if the direction of the impulse at $B$ bisect the angle $ABC$, the triangle $CBc$ is

isosceles, and $BC = Bc = AB$: therefore, the velocities in $BC$ and $AB$ are equal, the effect of the impulse has been therefore to change the direction without altering the velocity of the body.

Hence, the regular polygon inscribed in a circle center $S$, can be described with uniform velocity under the action of impulses tending to the center; and, by similar triangles $SBC$, $CBc$,

$$Cc : BC :: BC : BS.$$

And, if $V$ be the uniform velocity in the polygon, $T$ the time in a side $BC$, $BC = V.T$;

$$\therefore \; Cc = \frac{V^2 T^2}{BS}.$$

If now the number of sides be indefinitely increased, $Cc$ is ultimately twice the space through which the body is drawn from the tangent by the continuous force;

$\therefore \; \frac{2Cc}{T^2} = \frac{V^2}{BS}$ is the measure of the accelerating effect of the centripetal force.

*Illustrations of Circular Motion.*

1. In order to illustrate the action of a central force, we will suppose a small body attached by an inelastic string to a point on a smooth horizontal table.

If the body be set in motion by a blow perpendicular to the string, the string will remain constantly stretched, and the only force which acts on the body in the horizontal plane being in the direction of the fixed point, the areas described round this point are proportional to the time, and the body moves in a circle with uniform velocity.

If $v$ be the velocity of projection,

$l$ the length of the string,

the accelerating effect of the tension of the string is $\frac{v^2}{l}$;

*i. e.* $\frac{v^2}{l}$ is the velocity which would be generated from rest by the action of this tension continued uniformly, and the tension of the string

$$: \text{weight of the body} :: \frac{v^2}{l} : g.$$

Ex. If a velocity of two feet a second be communicated perpendicular to a string whose length is a yard,

$$v^2 : lg :: 4 : 3 \times 32 :: 1 : 24,$$

and the tension resulting is $\frac{1}{24}$th of the weight,

the time of revolution is evidently $\frac{2\pi l}{v}$ seconds $= \frac{6\pi''}{2} = 3 \times \frac{22''}{7}$, nearly,

$$= \frac{66''}{7} = 9''.43, \text{ nearly.}$$

2. If instead of being on a smooth table, the body moves on a rough table without rolling, the forces which act upon it will be the tension of the string in the direction of the point of attachment, and a force which is proportional to the weight of the body $= \mu \times$ weight, acting in the direction opposite to the motion, or in the tangent to the circular path.

Let $v$ be the velocity at the time $t$ in seconds,
$l$ the length of the string in feet.

In a short time $\tau$ the velocity being supposed to change from $v$ to $v'$, the accelerating effect of the tension is between $\frac{v^2}{l}$ and $\frac{v'^2}{l}$, and when $\tau$ is taken indefinitely small, these are equal, $\therefore \frac{v^2}{l}$ is the accelerating effect of the tension at the time $t$.

The change of the velocity is due to the friction whose retarding effect is measured by $\mu g$;

$\therefore$, if $V$ be the velocity of projection,

$$v = V - \mu g t;$$

and the body is reduced to rest in a time $\frac{V}{\mu g}$ seconds, after describing an arc $= \frac{V^2}{2\mu g}$ feet.

The tension $\propto \left(\frac{V}{\mu g} - t\right)^2$

$\propto$ (time)$^2$ from the instant of coming to rest.

3. Supposing that the Moon describes a circle with uniform velocity about the center of the Earth as its center, to find the ratio of the centripetal force exerted on the Moon to gravity at the Earth's surface.

Let $n$ = number of seconds in the Moon's periodic time, $R$ = the radius of the Moon's orbit in feet;

$$\therefore \frac{2\pi R}{n} = \text{velocity of the Moon,}$$

and $\frac{1}{R}\cdot\left(\frac{2\pi R}{n}\right)^2$ = measure of the accelerating effect of the force exerted on the Moon, and the measure of the same for gravity at the Earth's surface $= 32.2 = g$;

$\therefore$ force on the Moon : force of gravity at the Earth's surface

$$:: \frac{4\pi^2 R}{n^2} : g.$$

## XI.

1. If the sixth power of the velocity in circles be inversely proportional to the square of the periodic time, shew that the law of force varies inversely as the square of the radii.

2. Given the Earth's radius, the force of gravity at the Earth's surface, and the periodic time of the Moon, supposed to describe a circular orbit about the Earth, find her distance from the Earth's center.

3. Compare the areas described in the same time by the planets supposed to move in circular orbits about the Sun in the center exerting a force which $\propto$ (dist.)$^{-2}$.

4. A particle moves uniformly in a horizontal circle by a string, one yard long, fastened to the center of a circle, and

makes three revolutions in a second. Compare the tension of the string with the weight of the particle, the measure of gravity being 32.2.

5. If the force by which particles describe circles with uniform velocity varies as the distance, shew that the times of revolution are the same for all.

6. If the velocity of the Earth's motion were so altered that bodies had no weight at the equator, find approximately the alteration in the length of a day, assuming that the centripetal force at the equator is to its gravity :: 1 : 288.

7. A body moves in a circular groove under a force to the center, and the pressure on the groove is double the given force on the body to the center, find the velocity of the body.

8. If $F$ be the measure of the acceleration of a force which tends to a given center, and a body be projected from a point at a distance $R$ from the center at right angles to this distance with velocity $V$, and $V^2 = F.R$, shew that the body will describe a circle.

9. If a locomotive be passing a curve at the rate of twenty-four miles an hour, and the radius of the curve be $\frac{1}{16}$ of a mile, the resistance of the forces which retain it on the line, viz. the action of the rails on the flanges of the wheels, and the horizontal part of the forces which act perpendicular to the inclined road-way, being $R$, shew that $R = \frac{1}{100}$ of weight of the locomotive nearly.

10. If a body be attached by an extensible string to a fixed point in a smooth horizontal table, to find the velocity with which the body must move in order to keep the string constantly stretched to double its length.

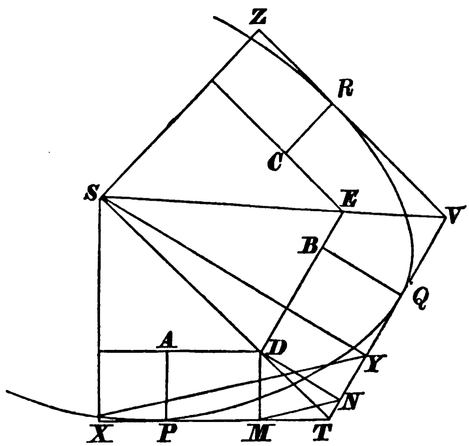
If $W$ be the weight which if suspended at the extremity of the string would just double its length, $P$ the weight of the body, $l$ the length of the string, shew that the square of the required velocity

$$= 2lg \cdot \frac{W}{P}.$$

PROP. V. PROBLEM I.

*Having given the velocity with which a body is moving at any three points of a given orbit, described by it under the action of forces tending to a common center, to find that center.*

Let the three straight lines $PT$, $TQV$, $VR$, touch the given orbit in the points $P$, $Q$, $R$ respectively; and let them meet in $T$ and $V$.




Draw $PA$, $QB$, $RC$ perpendicular to the tangents and inversely proportional to the velocities of the body at the points $P$, $Q$, $R$, i.e. such that

$$PA : QB :: \text{vel}^{y}.\text{ at } Q : \text{vel}^{y}.\text{ at } P,$$
$$QB : QC :: \text{vel}^{y}.\text{ at } R : \text{vel}^{y}.\text{ at } Q.$$

Through $A$, $B$, $C$ draw $AD$, $DBE$, $CE$ at right angles to $PA$, $QB$, $RC$ meeting in $D$ and $E$. Join $TD$, $VE$; $TD$ and $VE$ produced if necessary shall meet in $S$ the required center of force.

For, the perpendiculars $SX$, $SY$, let fall from $S$ on the tangents $PT$, $QTV$, are inversely proportional to the velo-

cities at $P$, $Q$, (Prop. I. Cor. 1) and are therefore directly as the perpendiculars $AP$, $BQ$, or as the perpendiculars $DM$, $DN$ on the tangents. Join $XY$, $MN$, and since $SX : SY :: DM : DN$ and the angles $XSY$, $MDN$ are equal; therefore, the triangles $SXY$, $DMN$ are similar;

$$\therefore SX : DM :: XY : MN,$$
$$:: XT : MT,$$

and the angles $SXT$, $DMT$ are right angles; therefore, $S$, $D$, $T$ are in the same straight line.

Similarly $S$, $E$, $V$ are in the same straight ine, and therefore, the center $S$ is in the point of intersection of $TD$, $VE$. Q. E. D.

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## XII.

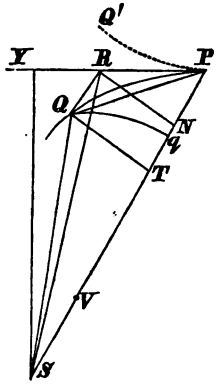
1. If $AB$, $BC$, $CD$ the three sides of a rectangle be the directions of the motion of a body at three points of a central orbit, and the velocities are proportional to these sides respectively, prove that the center of force is in the intersection of the diagonals of the rectangle.

2. If the velocities at three points of a central orbit be respectively proportional to the opposite sides of the triangle of which they are the angular points, the center of force is the center of gravity of the triangle.

### Prop. VI. Theorem V.

*If a body revolve about a fixed center of force, in any orbit whatever, in a non-resisting medium, and if at the extremity of a very small arc, commencing from any point in the orbit, a subtense of the angle of contact at that point be drawn parallel to the radius from that point to the center of force, then the force at that point tending to the center is ultimately as the subtense directly and the square of the time of describing the arc inversely.*

Let $PQ$ be the small arc, $PS$ the radius drawn from $P$ to $S$, the center of force. $RQ$ the subtense of the angle of contact at $P$ parallel to $PS$. $T$ the time of describing $PQ$. $F$ the accelerating effect of the force at $P$.




Then, when the body leaves $P$ it would, if not acted on by the central force, move in the direction $PR$, and if the force $F$ continued constant in magnitude and direction throughout the time $T$, $QR$ is ultimately the space through which it would have been drawn by $F$ in that time;

$$\therefore\ F = \frac{2QR}{T^2} \propto \frac{QR}{T^2}, \text{ ultimately.}$$

COR. 1. Draw $QT$ perpendicular to $SP$.

$$\text{Then, } F = \frac{2h^2}{SP^2} \cdot \frac{QR}{QT^2}, \text{ ultimately,}$$

where $h$ = twice the area described in an unit of time.

For area $PSQ = \frac{1}{2}hT$, (Prop. I.),

and since triangle $PSQ = \frac{1}{2}SP \,.\, QT$;

and area $PSQ$ = triangle $PSQ$, ultimately, (Lemma VIII.);

$$\therefore\ hT = SP \,.\, QT, \text{ ultimately;}$$

$$\therefore\ F = 2\frac{QR}{T^2} = \frac{2h^2 \,.\, QR}{SP^2 \,.\, QT^2}, \text{ ultimately.}$$

COR. 2. Draw $SY$ perpendicular on $PR$.

$$\text{Then } F = \frac{2h^2}{SY^2} \cdot \frac{QR}{PQ^2}, \text{ ultimately.}$$

For triangle $PSQ$ = triangle $PSR = \frac{1}{2} SY \,.\, PR$;

$$\therefore\ hT = SY \,.\, PR = SY \,.\, PQ, \text{ ultimately};$$

$$\therefore\ F = 2\frac{QR}{T^2} = \frac{2h^2}{SY^2} \,.\, \frac{QR}{PQ^2}, \text{ ultimately.}$$

Cor. 3. If the orbit have finite curvature at $P$, and $PV$ be the chord of the circle of curvature whose direction passes through $S$,

$$PV \,.\, QR = PQ^2, \text{ ultimately};$$

$$\therefore\ F = \frac{2h^2}{SY^2 \,.\, PV}.$$

Cor. 4. If $V$ be the velocity at $P$, then (Prop. I. Cor. 1, Art. 94),

$$V^2 = \frac{h^2}{SY^2} = 2F \,.\, \frac{PV}{4},$$

or the velocity at any point of a central orbit at which the curvature is finite, is that which would be acquired by a body moving from rest under the action of the central force at that point continued constant, after passing through a space equal to a quarter of the chord of curvature at that point drawn in direction of the center of force.

Cor. 5. Hence, if the form of any curve be given, and the position of any point $S$, towards which a centripetal force is continually directed, the law of the centripetal force can be found, by which a body will be deflected from its direction of motion, so as to remain in the curve. Examples of this investigation will be given in the following problems.

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*Observations on the Proposition.*

107. In Newton's enunciation of the proposition the sagitta of the arc, which bisects the chord and is drawn in the direction

of the center of force, is employed instead of the subtense used in the text, but it is easily seen that these are ultimately proportional, by reference to Art. 85.

The variations by which Newton expresses the results of the corollaries, are replaced by equations, in order to facilitate the comparison of the motion of bodies in different orbits and the forces acting upon them.

108. The figure employed in proof of the proposition is drawn upon supposition that the force is attractive, the orbit being concave to the center of force; the same proof applies also to the case of a repulsive force, if the curve be drawn in the direction of the dotted line $PQ'$ and the same construction be made.

The exception however should be made that the method fails in the particular positions in which the body is at the points of contact of tangents drawn from the center of force to the curve; in such cases $QR$ does not ultimately meet the tangent at a finite angle or is not a subtense, the result of the proposition is therefore not demonstrated for these particular positions; for a further description of the case see the note Art. 120, on the next proposition.

109. In the proof it is assumed that the body moves ultimately in the same manner as if the force $P$ remained constant in magnitude and direction, in which case the body would describe a parabola whose axis is parallel to $PS$, and which is evidently the parabola which has at $P$ the same curvature as the curve.

110. Hence the proposition contained in Cor. 4, can be more easily proved.

For, since the body moves in a parabola under the action of a constant force in parallel lines, the velocity at $P$ is that acquired by falling from the directrix under the action of the force

at $P$, continued constant, i.e. through a space equal to the distance of the focus of the parabola, which is equal to $\frac{1}{4}$ of the chord of curvature.

111. The supposition that the force at $P$ continued constant in magnitude and direction, causes the body to move in a curve which is ultimately coincident with the path of the body, may be justified by considering that if $PQ'$ be the arc of the parabola described on this supposition in the same time as the arc $PQ$ actually described, the error $Q'Q$ is due to the change in the magnitude of the forces and the direction of their action in the two cases; and the greatest difference of magnitude varies as the difference of $SP$ and $SQ$ ultimately, and the ratio of the error from this cause to $Q'R$ vanishes ultimately; also, since $\angle PSQ$ vanishes ultimately, the ratio of the error arising from the change of direction to $Q'R$ vanishes; therefore, $Q'Q : Q'R$ vanishes, or the curves may be considered ultimately coincident.

112. It is evident that the results of the Proposition and of the fourth corollary are true of the resultant of any force under the action of which any plane orbit is described, for this resultant may be supposed ultimately constant in direction and magnitude, in which case the curve described is a parabola: and the velocity at $P$ is that acquired by falling from the directrix whose distance is $\frac{1}{4}$ of the chord of curvature in direction of that resultant force.

Hence, in this case also,

$$V^2 = 2F \cdot \frac{PV}{4}, \text{ and } F = 2 \text{ limit } \frac{QR}{T^2}.$$

*Homogeneity.*

113. Cor. 1, 2. In the expressions for $F$ in these corollaries, it is of great importance to observe the dimensions of the symbols.

$h$, being an area, is of two dimensions;

$\therefore$ $h^2 . QR$ is of five, and $SP^2 . QT^2$ of four dimensions;

$$\therefore \frac{2h^2 . QR}{SP^2 . QT^2} \text{ is of one dimension,}$$

and represents the line equal to twice the space through which a force would draw a body in an unit of time, or the space which represents the velocity generated by the force in an unit of time, either of which may be taken as the measure of the accelerating effect of the force.

Hence, if the actual areas, lines, &c. be represented by the symbols, and not the *number* of units, as mentioned in Art. 103, every term of an equation or of a sum or difference must be homogeneous or of the same number of dimensions; for example, the addition of an area and a line can have no interpretation.

*Tangential and Normal Forces.*

114. *To find the accelerating effect of the components of the forces under the action of which a body describes any plane curve, in the directions of the normal and tangent at any point.*

Let $PQ$ be a small arc of the curve described under the action of any forces, $F$, $G$ the measures of the accelerating effect of these forces in the direction of the tangent and perpendicular to it. Then, if $V$ be the velocity at $P$, $T$ the time of describing $PQ$, the forces may be supposed ultimately to remain the same in magnitude and direction, and if $QR$ be perpendicular to $PR$, we have ultimately

$$PR = V . T + \tfrac{1}{2} F . T^2, \text{ and } QR = \tfrac{1}{2} G . T^2,$$

and the ratio of $F . T^2 : V . T$ vanishes ultimately;

$$\therefore \frac{PR^2}{QR} = \frac{2V^2}{G}, \text{ ultimately;}$$

and if $R$ be the radius of curvature at $P$, $\frac{PR^2}{QR} = 2R$ ultimately;

$$\therefore\ G = \frac{V^2}{R}.$$

Also if $PU = V.T$ be measured in $PR$, $UR$ is the space described under the action of the tangential component ultimately;

$$\therefore\ F = \frac{2UR}{T^2} = \frac{2(PR - V.T)}{T^2}, \text{ ultimately},$$

$$= \frac{2(PQ - V.T)}{T^2}, \text{ ultimately}.$$

Also, if $V'$ be the velocity at $Q$ since the velocity is ultimately the component of the velocities whose squares are

$$V^2 + 2F.PR \text{ parallel to } PR \text{ and } 2G.QR \text{ in } QR;$$

$$\therefore\ V'^2 = V^2 + 2F.PR + 2G.QR, \text{ ultimately},$$

and $QR : PR$ vanishes ultimately;

$$\therefore\ F = \frac{V'^2 - V^2}{2PQ}, \text{ ultimately}.$$

Again, $PQ = V.T = \frac{1}{2}(V + V')T$, ultimately;

$$\therefore\ F = \frac{V' - V}{T}, \text{ ultimately}.$$

115. *To find the velocity at any point of an orbit described under the action of any forces in one plane.*

Let $AB$ be any arc of an orbit, $V$, $v$ the velocities at $A$ and $B$, and suppose the arc $AB$ divided into a large number of small portions of which $PQ$ is one, $v_r$ $v_{r+1}$ velocities at $P$ and $Q$, $F$ the accelerating effect of the tangential component of the forces at $P$,

$$v_{r+1}^2 - v_r^2 = 2F.PQ,$$

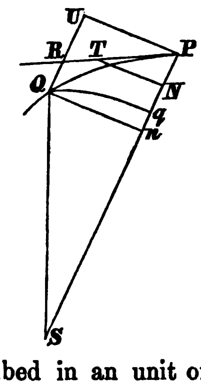
and $v^2 - V^2$ is obtained by taking the limit of the sum of the magnitudes $2F \cdot PQ$ corresponding to the different arcs when their number is indefinitely increased.

That this is rigidly correct may be shewn by considering that $v_{r+1}^2 - v_r^2 : 2F.PQ$ is ultimately a ratio of equality; therefore, by Lemma IV, Art. 23, the limiting ratio of the sums is also a ratio of equality.

*Radial and Transverse Forces.*

116. *To find the accelerating effect of the components of forces under the action of which a body describes any plane curve, in the direction of a line passing through a fixed point, and perpendicular to it.*

Let $PQ$ be a small arc described in the time $T$, $P$, $Q$ the measure of the accelerating effect of the components in $PS$, and $PU$ perpendicular to $PS$; $PR$ a tangent at $P$, $QRU$, $PU$ parallel and perpendicular to $SP$. $V$ the velocity at $P$, $PT = V \cdot T$, $TN$ perpendicular to $SP$, $Qq$ the arc of a circle, center $S$.




Since the forces may be considered ultimately constant in magnitude and direction,

$$\tfrac{1}{2} P \cdot T^2 = Nn = Nq + \frac{Qn^2}{2Sq}, \text{ ultimately.}$$

Let $h = 2 \times$ area which would be described in an unit of time by radii from $S$ if the transverse force at $P$ ceased to act,

$$Qn \cdot SP = TN \cdot SP = hT, \text{ ultimately,}$$

and if $P'$ be the measure of the accelerating effect of a force under the action of which the body would move in $PS$ so that its distance from $S$ would be always equal to that of the body in $PQ$ at the same time, $\tfrac{1}{2} P' \cdot T^2 = Nq$,

$$\text{and } \frac{Qn^2}{2Sq} = \frac{h^2T^2}{2SP^3}, \text{ ultimately;}$$

$$\therefore\ P = P' + \frac{h^2}{SP^3}.$$

$$\text{Again, } \tfrac{1}{2}\,Q\,.\,T^2 = Qn - TN,$$

and if at $Q$ $h'$ corresponds to $h$, $h' - h$ the increase of $h$ is due to the increase of velocity in direction $PU$, which is equal to

$$\frac{2(PU - TN)}{T};$$

$$\therefore\ h' - h = \frac{2(PU - TN)}{T}\,SP, \text{ ultimately;}$$

$$\therefore\ Q = \frac{h' - h}{SP\,.\,T}, \text{ ultimately.}$$

*Angular Velocity.*

117. Def. *Angular velocity* about a fixed point is uniform, when equal angles are described in equal times by radii drawn to the fixed point.

*Uniform angular velocity* is measured by the angle described in an unit of time.

*Variable angular velocity* is measured by the angle which would be described by a radius in an unit of time, if moving with uniform angular velocity equal to the angular velocity at the time under consideration; this is the limit of the angle described in a time $T$ divided by $T$, when $T$ is indefinitely diminished; for, let $PSQ$ be the angle described about $S$ in a time $T$, then, since this may be ultimately supposed to be described uniformly with the angular velocity at $P$,

$\therefore$ the angular velocity at $P \times T = \angle PSQ$, ultimately.

118. *To find the angular velocity in a central orbit.*

Let $PQ$ be a small arc described in the time $T$, draw $QT$ perpendicular to $SP$, and let $h$ = twice the area described in an unit of time.

$hT$ = twice the area $PSQ = QT \cdot SP$ ultimately, and the angles being supposed estimated in circular measure,

$$\frac{QT}{SQ} = \text{angle } PSQ, \text{ ultimately};$$

$$\therefore\ hT = SP \cdot SQ \times \text{angle } PSQ, \text{ ultimately};$$

$$\therefore\ \text{angular velocity} = \text{limit of } \frac{\text{angle } PSQ}{T}$$

$$= \frac{h}{SP^2}.$$

119. *To find the angular velocity of* SY.

Let $PV$ be the chord of curvature through $S$.

The angle described by $SY$ in the time $T$

= angle between the tangents at $P$ and $Q$

$$= 2 \text{ angle } PVQ = \frac{2QT}{QV} \text{ ultimately},$$

$$\text{and } \angle PSQ = \frac{QT}{SQ} \text{ ultimately},$$

$\therefore$ angular vel. of $SY$ : angular vel. of $SP$ :: $\frac{2QT}{QV} : \frac{2QT}{QS}$ ultimately $= 2SP : PV$;

$$\therefore\ \text{angular vel. of } SY = \frac{2h}{PV \cdot SP}.$$

*Illustrations.*

1. Find the force under the action of which a body may describe the equiangular spiral uniformly.

The velocity being constant there is only a normal force measured by $(\text{vel.})^2 \div \text{radius of curvature} = \frac{V^2 \sin\alpha}{SP}$.

2. Find the force tending to the pole of the cardioid, under the action of which the curve is described.

$$(\text{vel.})^2 = \frac{h^2}{SY^2} = \frac{h^2 . BC}{SP^3},$$

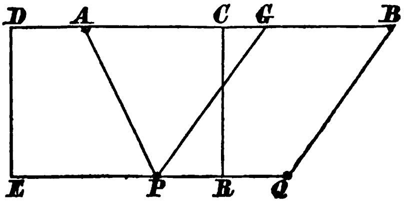
$$PV = \frac{4}{3} SP;$$

$\therefore$ the accelerating effect of the force is $\dfrac{3h^2 . BC}{2SP^4} \propto \dfrac{1}{SP^4}$.

3. Two rings $P$, $Q$ slide on a string which passes round two fixed pegs $A$, $B$ in a smooth horizontal plane; the rings are brought together, and then projected with equal velocities, so as to keep the string stretched: shew that the tension of the string varies inversely as the distance $AP$.

Let the figure represent the position of the strings at any time.

Let $CR$ bisect $AB$ and $PQ$, and let $DE$ be parallel to $CR$ so that $EP = PA$, then $EPR = AP + PR$ is constant;




$\therefore$ $P$ moves in a parabola whose focus is $A$, directrix $DE$, and the tensions of the string in $PA$, $PQ$ being equal and equally inclined to the tangent to $P$'s path, the resultant of these tensions, which are the only forces acting in the plane of the curve, is in the normal, and if $T$ be the measure of the accelerating effect of the tension, $PG$ the normal, $R$ the radius of curvature,

$$2T \cos APG = \frac{V^2}{R},$$

and $2R \cos APG =$ chord of curvature through $A$

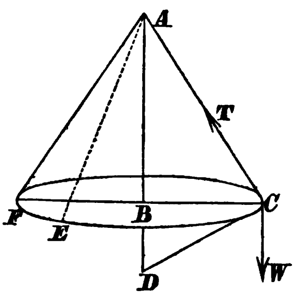
$$= 4PA,$$

$$\therefore T = \frac{V^2}{4PA} \propto \frac{1}{PA},$$

since $V$ is constant.

4. A body is suspended by a string to a fixed point, and being drawn out of the vertical is projected horizontally so as to describe a horizontal circle with uniform velocity. Find the velocity and tension.

Let $A$ be the point of suspension, $BC$ the radius of the circle described; $\therefore$ the circle being described uniformly, the resultant force on the body tends to the center $B$, and the measure of the accelerating effect of the resultant force is




$$\frac{V^2}{BC} \text{ in direction } CB.$$

Let $T$, $W$ be the tension of the string, and the weight of the body acting in $CA$ and parallel to $AB$;

$$\therefore \frac{V^2}{BC} : g :: CB : AB;$$

$$\therefore V^2 = \frac{g \cdot BC^2}{AB},$$

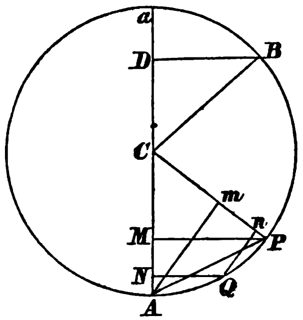
and if $CD$ be perpendicular to $AC$, $BC^2 = AB \cdot BD$;

$\therefore$ the velocity is that due to the space $\frac{1}{2} BD$,

and $T : W :: CA : AB$.

5. A body revolves in a smooth circular tube under the action of a force tending to any point in the circumference, and varying as the distance from that point. Find the pressure on the tube and the point where there is no pressure, the motion commencing from a given point.

Take $A$ the center of force, $C$ that of the circle, $B$ the point of starting, $PQ$ a small arc, $BD$, $PM$ $QN$ ordinates, $Am$, $Qn$ perpendicular on $CP$, $\mu$ $PA$ the measure of the accelerating effect of the




force, $\therefore \mu . mA$, $\mu . Pm$ are those of the tangential and normal forces,

$= \mu . PM$ and $\mu . AM$ respectively,

$(\text{vel.})^2$ at $Q - (\text{vel.})^2$ at $P = 2\mu . PM . PQ$

$= 2\,\mu . CP . MN$, ultimately,

whence $(\text{vel.})^2$ at $P = 2\mu CP . DM$,

and $\dfrac{(\text{vel.})^2 \text{ at } P}{CP} = \mu . AM \pm$ accelerating effect of the pressure on the tube;

$\therefore$ pressure on the tube has for the measure of its accelerating effect

$$\mu\,(2DM \sim AM) = \mu\,(2AD \sim 3AM);$$

$\therefore$ the pressure is outwards from $B$ until $AM = \frac{2}{3} AD$, at which point there is no pressure, and inwards from that point to the corresponding one on the opposite side, having its greatest value at $A$, and the outward pressure at $B$ is half the inward pressure at $A$.

6. To find the tension of a string, by which a body is attached to the center of a vertical circle in which it revolves.

Let $P$ be the position of the body at any time, $u$ the velocity at $A$ the lowest point, $CP$ the radius of the circle,

$$(\text{vel.})^2 \text{ at } P = u^2 - 2g . AM,$$

and the accelerating effect of the tension of the string is measured by

$$\frac{u^2 - 2g . AM}{CA} + \frac{g . CM}{CP};$$

$$\therefore \text{ the tension of the string : weight of the body}$$
$$:: u^2 - 2g . CA + 3g . CM : g . CA.$$

Cor. 1. In order that the complete circle may be described, since the string must be stretched at the highest point where $-CA$ must be written for $CM$, $u^2 =$ or $> 5g . CA$, and if the circle be just described the tension at the lowest point $= 6 \times$ weight.

Cor. 2. If the body oscillates the extent of the oscillation is given by the consideration that at the extremity of the arc of oscillation

$$u^2 = 2g . AM,$$

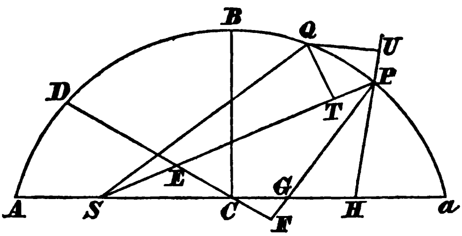
and $AM$ is less than $AC$, otherwise the string would not be stretched

$\therefore$ in this case the tension of the string at $A$

$$=\frac{2AM+AC}{AC}\times \text{weight of the body.}$$

7. If in a smooth elliptic tube a particle be placed at any point and be acted on by two forces which tend to the foci, and vary inversely as the square of the distances from those points, to shew that the pressure at any point varies inversely as the radius of curvature.

Let $O$ be the point of starting, $PQ$ a small arc described by the body,




$QT$, $QU$ perpendiculars on $SP, HP$; $\frac{\mu}{SP^2}$, $\frac{\mu'}{HP^2}$, $R$ the measures of the accelerating effects of the forces, and the pressure of tube. Then, employing the usual letters for the lines of the figure, the accelerating effect of the tangential component of force to $S$ is

$$\frac{\mu}{SP^2}\cdot\frac{PT}{PQ}=\frac{\mu(SP-SQ)}{SP.SQ.PQ}=\frac{\mu}{PQ.SQ}-\frac{\mu}{PQ.SP}, \text{ ultimately;}$$

$$\therefore\ (\text{vel.})^2 \text{ at } Q-(\text{vel.})^2 \text{ at } P=\frac{2\mu}{SQ}-\frac{2\mu}{SP}-\left(\frac{2\mu'}{HP}-\frac{2\mu'}{HQ}\right) \text{ (Art. 115);}$$

$$\therefore\ (\text{vel.})^2 \text{ at } P=\frac{2\mu}{SP}-\frac{2\mu}{SO}-\frac{2\mu'}{HO}+\frac{2\mu'}{HP}.$$

$$\text{Also } \frac{(\text{vel.})^2 \text{ at } P}{\rho}=\frac{PF}{PE}\left(\frac{\mu}{SP^2}+\frac{\mu'}{HP^2}\right)-R,$$

if $\rho$ be the radius of curvature at $P$,

$$\text{and } 2\rho\,.\frac{PF}{PE}=PV=\frac{CD^2}{AC}=\frac{SP.HP}{AC};$$

$$\therefore R.\rho = \frac{\mu . HP}{AC.SP} + \frac{\mu'.SP}{AC.HP} - \frac{2\mu'}{SP} - \frac{2\mu'}{HP} + \frac{2\mu}{SO} + \frac{2\mu'}{HO}$$

$$= \frac{2\mu}{SO} + \frac{2\mu'}{HO} - \frac{\mu}{AC} - \frac{\mu'}{AC} = \frac{\mu HO}{SO} + \frac{\mu' SO}{HO};$$

$$\therefore R \propto \frac{1}{\rho}.$$

## XIII.

1. Two equal bodies lie on a rough horizontal table, and are connected by a string, which passes through a fine ring on the table; if the string be stretched, find the greatest velocity with which one of the bodies can be projected in a direction perpendicular to its portion of the string without moving the other body.

2. If a body be attached to a point by a thread, and be projected so as to describe a vertical circle, prove that if $T_1$, $T_2$ be the tensions of the string at two points in any diameter, the arithmetic mean between $T_1$, $T_2$ is independent of the position of the diameter and that $T_2 \sim T_1$ is six times the component of the weight in the direction of the diameter.

3. A string of given length $l$ is capable of sustaining a weight $W$. One end of it being fixed and a given weight $W'$ ($< W$) being attached to the other end, it revolves with a given angular velocity on a smooth horizontal plane through the fixed point. Find the greatest tangential impulse that may be applied to $W'$ without breaking the string.

4. In the case of Problem 3, if $W'$ oscillate in a vertical plane, find the greatest arc through which the body can oscillate without breaking the string.

5. A body slides down a smooth cycloidal arc, whose axis is vertical, find the pressure at any point of the cycloid, and shew that if it fall from the highest point, the pressure at the lowest point is twice the weight of the body.

6. A particle moves in a circular tube, under the action of a force which tends to a point in the tube, and whose accelerating effect varies as the distance, shew that, if the particle begins to move from a point at a distance from the center of force equal to the radius, there is no pressure on the tube at an angular distance from the center of force equal to $\cos^{-1}\frac{2}{3}$.

7. A ring slides on a string hanging over two pegs in the same horizontal line, find the tension of the string at the lowest point if the ring begin to fall from the point in the horizontal line through the pegs, the string being stretched.

8. In a central orbit, shew that the centripetal force is to the force which would cause it to approach directly with its paracentric velocity in the orbit, as $2SP^3 : 2SP^3 - SY^2 . PV$.

9. A curve is described by a body under the action of a central force the measure of whose accelerating effect is $\frac{\mu}{SP}$, prove that the angular velocity of the perpendicular on the tangent : that of the radius vector :: $\mu$ : $V^2$.

10. Determine the force under the action of which an equiangular spiral may be described with uniform velocity.

11. A body describes a complete circle about a fixed point $C$ to which it is attached by a thread, and is at the same time attracted to a fixed center $S$ of force $\propto$ (dist.)$^{-2}$ in the plane of the circle, find the least possible velocity of projection from the point where the circle meets $CS$ or $CS$ produced.

If $S$ bisect a radius, shew that this velocity : velocity when $S$ is at $C$ :: 28 : 9.

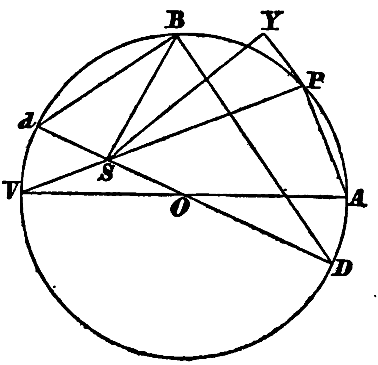
12. A particle moves in a smooth elliptic groove, under the action of two forces tending to the foci and varying inversely as the squares of the distances, being equal at equal distances. Prove that, if the velocity at the extremity of the axis major be to that at the extremity of the axis minor as $AC$ to $BC$, then the

velocity at any point varies inversely as the normal: and find the pressure on the tube.

## Prop. VII. Problem II.

*A body moves in the circumference of a circle, to find the law af the centripetal force tending to any given point in the plane of the circle.*

Let $APV$ be the circumference of the circle, $S$ the given point to which the centripetal force tends, $PV$ the chord of the circle drawn through $S$ from $P$ the position of the body at any time.




Let $SY$ be drawn perpendicular to the tangent $PY$ at $P$.

By Prop. VI. Cor. 3, if $F$ be the measure of the accelerating effect of the centripetal force,

$$F = \frac{2h^2}{SY^2 . PV},$$

and since the angles $SPY$, $VAP$ are equal and also the right angles $PYS$, $APV$, the triangles $SPY$, $VAP$ are similar;

$$\therefore SY : SP :: PV : VA;$$

$$\therefore F = \frac{2h^2 . VA^2}{SP^2 . PV^3};$$

$$\therefore F \text{ varies inversely as } SP^2 . PV^3,$$

since $h$ and $VA$ are given.

Cor. 1. Hence, if the given point $S$ to which the centripetal force tends, be situated on the circumference of

the circle, $V$ coincides with $S$, and $F$ varies inversely as $SP^5$.

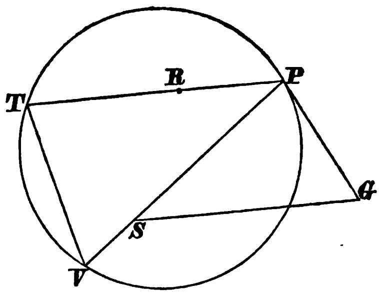
COR. 2. The force, under the action of which a body $P$ revolves in a circle $APTV$, is to the force, under the action of which the same body $P$ can revolve in the same circle in the same periodic time about any other center of force $R$, as $RP^2 . SP$ to $SG^3$, $SG$ being a straight line drawn from the first center $S$ parallel to the distance $RP$ of the body from the second center of force $R$, to meet $PG$, a tangent to the circle.

For, by the construction of this proposition, since the periodic times are the same, the areas described in a given time are the same; therefore, $h$ is the same for both centers, hence, if $PRT$ be the chord through $R$, the force tending to $S$ : force tending to $R$

$:: RP^2 . PT^3 : SP^2 . PV^3$;

but, by similar triangles $TPV$, $GSP$,

$PT : PV :: SP : SG$;




$$\therefore \text{ force tending to } S : \text{force tending to } R$$
$$:: RP^2 . SP^3 : SP^2 . SG^3$$
$$:: RP^2 . SP : SG^3.$$

COR. 3. The force under the action of which a body $P$ revolves in any orbit about a center of force $S$, is to the force under the action of which the same body $P$ can revolve in the same orbit in the same periodic time about any other center of force $R$, as $RP^2 . SP$ to $SG^3$, $SG$ being the straight line drawn from the first center of force $S$, parallel to $RP$ the distance of $P$ from the

second center of force $R$, to meet $SG$ the tangent to the orbit.

For, in each case, the body may be supposed for a short time to be moving in the circle of curvature, and the forces are the same as those which would retain the body in the circular orbit; therefore, since the areas described in a given time are equal, the ratio of the forces is $RP^2 . SP : SG^3$.

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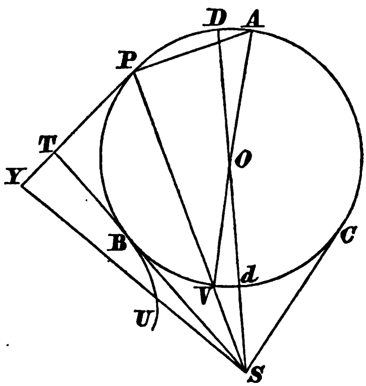
*Repulsive Forces.*

120. In the figure employed in the proposition, the force is supposed to be attractive, but the law of force is also given by the proposition, for the case in which the center of force $S$ is exterior to the circle, in which case the force is repulsive through the arc $BC$, which is convex to the center of force, and contained between the tangents from $S$.




It is important, however, to observe that this problem is to find what would be the law of force tending to $S$, under the action of which a body would be moving, *supposing that* it could move in a circle under the action of such a force, but it does not assert the possibility of such a motion.

In fact, the *complete* description of a circle $ABC$ under the sole action of a central force tending to an external point $S$ is impossible, because, as the body approaches the point $B$, the component of the velocity perpendicular to $SB$ remains finite however

near the body approaches $B$, and since there is no force to generate a velocity in the opposite direction, the body must proceed to describe an arc $BU$ on the opposite side. $SB$ would be a tangent to both curves, because the velocity in direction $BS$ becomes larger than any finite quantity, as the body approaches $B$, and therefore the angle between $BS$ and the direction of motion is indefinitely small at $B$.

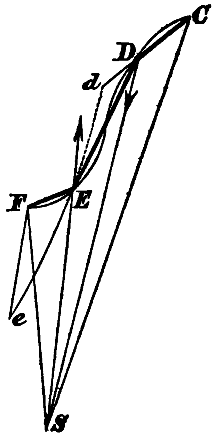
That a finite velocity in the direction perpendicular to $SB$ could remain up to $B$, may be shewn by producing $SB$ to $T$ in the tangent $PY$ at $P$: then the component of the velocity at $P$ perpendicular to $SB$ is

$$\frac{h}{SY} \cdot \frac{SY}{ST} = \frac{h}{ST} = \frac{h}{SB},$$

when the body arrives near $B$.

121. The force at a point indefinitely near to $B$ cannot be properly determined by the method of Prop. VI., because the lines parallel to the direction of the force from which the measures of the force are obtained are not subtense, or sagitta, being not inclined in this case at a finite angle to the tangent.

But it can be seen in another manner from the polygon of Prop. I. that the force is infinitely great when the distance from $B$ becomes infinitely small.




Thus, if $CDEF$ be a portion of the polygon whose limit touches the radius from $S$ between $D$ and $E$, the angle between $DE$ and $DS$ or $ES$ may be made as small as we please, hence the velocity generated by the impulse in the directions $DS$ and $SE$ becomes infinitely great compared with the velocities in $CD$ and $EF$.

In the figure the impulses at $D$ and $E$ are in the direction of the forces which

are their limits, being supposed attractive and repulsive respectively.

122. *If a circle be described by a body under the action of forces tending to a point in the circumference, the force varies inversely as the fifth power of the distance from that point, at all points at a finite distance from* S.

For, in this case,

$$PV = SP, \text{ and } SY : SP :: SP : SA;$$

$$\therefore F = \frac{2h^2}{SY^2 . PV} = \frac{2h^2}{SP^3} . \frac{SP^2}{SY^2} = \frac{2h^2 SA^2}{SP^5};$$

$$\therefore F \propto \frac{1}{SP^5}.$$



We may also observe here that the possibility of a description of a circle is not asserted, but only the law of force required in case of such a description. The body would pass to the other side of the tangent on arriving at $S$.

*Velocity in the Circular Orbit.*

123. *To find the velocity in the circular orbit described under the action of a force tending to any point in the plane of the orbit.*

$$\text{The velocity at } P = \frac{h}{SY} = \frac{h}{SP} . \frac{SP}{SY} = \frac{h}{SP} . \frac{VA}{PV}$$

$$\propto \frac{1}{SP . PV}.$$

COR. If $S$ be in the circumference of the circle, and $\frac{\mu}{SP^5}$ be the measure of the accelerating effect of the force,

$$\mu = 2h^2 SA^2;$$

$$\therefore \text{ velocity at } P = \frac{h \,.\, VA}{SP^2} = \frac{\left(\frac{\mu}{2}\right)^{\frac{1}{2}}}{SP^2},$$

or, we may employ the result of Cor. 4,

$$V^2 = F \,.\, \frac{PV}{2} = \frac{\mu}{SP^5} \,.\, \frac{SP}{2};$$

$$\therefore \; V = \left(\frac{\mu}{2}\right)^{\frac{1}{2}} \,.\, \frac{1}{SP^2} \propto \frac{1}{SP^2}.$$

*Absolute Force.*

124. If the force upon a body placed at any distance from the point $S$ varies inversely as the $n$th power of that distance, the magnitude of the force is determined, or its ratio to any given force as that of gravity, when the distance $SP$ is given. The measure of the accelerating effect of the force is written $\frac{\mu}{SP^n}$ where $\mu$ the constant part of this measure is an algebraical symbol of $n+1$ dimensions, $\frac{\mu}{SP^n}$ is the space which represents the velocity generated in a body in an unit of time by a constant force equal to the force acting on the body at $P$. If the unit of space $= a$, $\frac{\mu}{a^n}$ is the measure of the accelerating effect of the force on a body at an unit of distance, and $\mu$ is called the *Absolute Force*, being the measure of the accelerating effect of the force at an unit of distance $\times$ the $n$th power of that unit. The absolute force is not the measure of the accelerating effect of any force, unless the symbols be treated numerically, in which case $\mu$ is twice the number of units of space through which a force equal to that at an unit of distance would draw a body from rest in an unit of time.

*Law of Force in a Circular Orbit.*

125. The law of force may be expressed in terms of the distances $SP$, for $SD$, $Sd$ being the greatest and least distances of the body from $S$,

$$SD \,.\, Sd = SP \,.\, SV; \quad \therefore \; SP \,.\, PV = SP^2 \pm SD \,.\, Sd;$$

$$\therefore \; F = \frac{2h^2 \,.\, AV^2 \,.\, SP}{(SP^2 \pm SD \,.\, Sd)^3}$$

$$= \frac{2h^2 \,.\, AS^2}{SP^5},$$

if $Sd = 0$, or $S$ be on the circumference of the circle.

If $S$ be exterior to the circle, $SD \,.\, Sd = SB^2$, and the lower sign is taken;

$$\therefore \; F = \frac{2h^2 AV^2 \,.\, SP}{(SP^2 - SB^2)^3}.$$

*Periodic Time.*

126. *To find the periodic time in a circular orbit described under the action of a force tending to a point in the circumference.*

Let $P$ be the periodic time, $R$ the radius of the circle, $\frac{\mu}{SP^5}$ the measure of the accelerating effect of the force at $P$,

$$h \,.\, P = \text{twice the area of the circle} = 2\pi R^2,$$

$$\text{and } \mu = 2h^2 AS^2 = 8h^2 R^2;$$

$$\therefore \; P = \frac{4\sqrt{2}\,\pi R^3}{\mu^{\frac{1}{2}}}.$$

127. *To compare the periodic times in the same circle when described under the action of a force tending to a point in the circumference, and a force tending to the center of the*

*same magnitude as the force at a distance equal to the radius of the circle.*

Let $P'$ be the periodic time, and $V$ the uniform velocity in the circle in the second case,

$$V^2 = \frac{\mu}{R^5} . R ; \quad \therefore \; V = \frac{\mu^{\frac{1}{2}}}{R^2},$$

$$\text{and } P' . V = 2\pi R ; \quad \therefore \; P' = \frac{2\pi R^3}{\mu^{\frac{1}{2}}};$$

$$\therefore \; P = 2\sqrt{2} . P'.$$

*Illustrations.*

1. When the force in a circular orbit tends to a point within the circle, to find the point at which the true angular velocity is equal to the mean angular velocity.

The true angular velocity is measured by $\frac{h}{SP^2}$, the mean angular velocity by $\frac{2\pi}{P}$, if $P$ be the periodic time; but $h . P = 2\pi R^2$;

$$\therefore \text{ at the required point } \frac{h}{SP^2} = \frac{h}{R^2}, \text{ and } SP = R,$$

or the point is in the perpendicular to the distance $SC$ of $S$ from the center of the circle, bisecting that distance.

2. If the measures of the accelerating effect of the force at the greatest and least distances $SD$, $Sd$, of a body in its circular orbit from the point to which the force tends be the radius and twice the diameter respectively, the unit of time being a second, to find the number of seconds in passing from $D$ to $d$.

Since $\frac{8h^2R^2}{SD^2 . Dd^3} = R$, and $\frac{8h^2R^2}{Sd^2 . Dd^3} = 4R$;

$$\therefore \; SD = 2\,Sd, \text{ and } Dd = 3\,Sd,$$

and if $T$ = the number of seconds from $D$ to $d$,

$$h \,.\, T = \pi R^2, \text{ and } \frac{h^2}{Sd^2} = 4R^2;$$

$$\therefore\ h = 2R \,.\, Sd = \frac{4}{3} R^2;$$

$$\therefore\ T = \frac{3}{4} \,.\, \pi.$$

## XIV.

1. Compare the forces by which a body attracted separately to two centers of force may describe the same circle in different periodic times.

2. If $SB$ (fig. page 167,) be perpendicular to the diameter $DSd$, prove that the forces at $D$ and $d$ are as $dB^4 : DB^4$.

3. If $\mu$ be the absolute force in a circular orbit described under the action of a force tending to a point in the circumference, prove that the time in a quadrant commencing from the extremity of the diameter through the center of force is $(\pi + 2)\sqrt{\dfrac{2R^6}{\mu}}$. In what unit of time is the result expressed?

4. Prove that $\dfrac{V^3}{F}$ is finite however near the body approaches $B$ if the circular orbit be described about an external point.

5. Prove that, if the law of force tending to $S$, a point without a circle, be the law of force under which part of the circle can be described, the body will move near $B$ as if acted on by a force tending to $B$ and varying inversely as the cube of the distance from $B$.

Also give reasons for supposing that no force acts at $B$.

6. $OE$ is a radius perpendicular to the diameter through $S$ in a circular orbit about a central force to $S$, $SB$ an ordinate,

perpendicular to $OS$, shew that if the force at $B$ be an arithmetic mean between the forces at the greatest and least distances, $OE^3 = SB \cdot SE^2$.

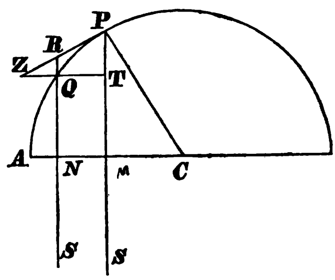
7. Apply the proposition contained in Cor. 3, to prove that if in an elliptic orbit described under the action of a force tending to the center, the force varies as the distance from the center, then the force tending to the focus varies inversely as the square of the focal distances.

8. Deduce by Cor. 3, the law of force by which a parabola described under the action of a force tending to the focus, from the constant force parallel to the axis under the action of which the parabola may be described.

## Prop. VIII. Problem III.

*A body moves in a semicircle* PQA *under the action of a force tending to a point* S *so distant that the lines* PS, QS *drawn from the body to that point may be considered parallel; to find the law of force.*

Let $CA$ be a semidiameter of the semicircle drawn from the center perpendicular to the direction in which the force acts, cutting $PS$, $QS$ in $M$ and $N$, and join $CP$.




Let $PRZ$ be the tangent at $P$, $ZQT$ perpendicular to $PMS$ meeting $PRZ$ in $Z$, and let $SNQ$ meet $PRZ$ in $R$.

Then the force at $P \propto \dfrac{QR}{SP^2 \cdot QT^2}$ ultimately, if the arc $PQ$ be nascent from $P$, and $SP$

may be considered constant; also by Euclid III. 36, $QR.(RN+QN)=RP^2$, and since $QR$ is parallel to $PT$ and the triangles $PZT$, $CPM$ are similar,

$$RP : QT :: PZ : ZT :: CP : PM;$$

$$\therefore \frac{QT^2}{QR}=\frac{QT^2}{RP^2}\cdot\frac{RP^2}{QR}=\frac{PM^2}{CP^2}\cdot(RN+QN)$$

$$=\frac{2PM^3}{CP^2}, \text{ ultimately};$$

$$\therefore \text{ force at } P \propto \frac{QR}{QT^2} \propto \frac{1}{PM^3}.$$

*Aliter.*

In fig. Art. 119, draw $OE$ a semidiameter perpendicular to $SD$, and let the distance $SP$ cut the circle in $V$, and $OE$ in $M$, then, by the preceding proposition,

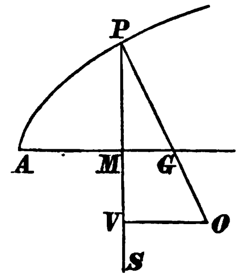
$$F=\frac{8h^2R^2}{SP^2.PV^3},$$

and if $S$ be very distant the ratio of $PM : SM$ or $SO$ vanishes; therefore, $SP=SO$ ultimately, and $PV$ is ultimately perpendicular to $OE$ and $=2PM$;

$$\therefore F=\frac{h^2R^2}{SO^2.PM^3} \propto \frac{1}{PM^3}.$$

SCHOLIUM.

A body moves in an ellipse, hyperbola or parabola, under the action of a force tending to a point so situated and so distant that the lines drawn from the body to that point may be considered parallel and perpendicular to the major axis of the ellipse, the axis of the parabola or the transverse axis of the hyperbola. To shew that the force varies inversely as the cube of the ordinates.




Let $AMG$ be the axis to which the direction of the forces may be considered perpendicular, $PM$, $PG$ the ordinate and normal, $PO$, $PV$ the diameter and chord of curvature in direction $PS$.

Then $F = \dfrac{2h^2}{SY^2 . PV} = \dfrac{2h^2}{SP^2 . PV} . \dfrac{PG^2}{PM^2}$,

since $SY : SP :: PM : PG$;

$$\therefore \; F \propto \frac{PG^2}{PM^2 . PV} \propto \frac{PG^3}{PM^3 . PO} \propto \frac{1}{PM^3},$$

since $PO \propto PG^3$, see Art. 80.

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*Observations on the Proposition.*

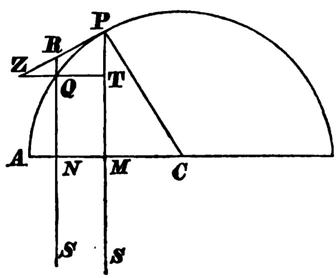
128. If $S$ be removed to an infinite distance $\dfrac{h^2}{SP^2}$ becomes the ratio of two infinite quantities, but it can be shewn to be a finite area; for if $u$ be the velocity perpendicular to the direction of the force, $h . T$ being twice the area $SPQ$ described in the time $T$ in which $PQ$ is described, and area $SPQ$ = area $SMN$, ultimately, $= \frac{1}{2} SC \times u . T$;

$$\therefore \; h = SC . u;$$

$$\therefore \; \frac{h^2}{SP^2} = \frac{h^2}{SC^2} = u^2 \text{ ultimately,}$$

and $F = \dfrac{2u^2 . QR}{QT^2}$ ultimately,

$= \dfrac{u^2 . CP^2}{PM^3}$ in the semicircle.



*Extension of Scholium.*

129. When a body describes any curve under the action of a force tending to a point $S$ so distant that the lines drawn from $S$ to the body may be considered parallel, to find the law of force.

Using the figure and construction for the Scholium, $AMG$ being any line perpendicular to the direction of the force,

$$F = \frac{2h^2}{SY^2 . PV} = \frac{2h^2}{SP^2 . PV} \cdot \frac{SP^2}{SY^2} = \frac{2h^2}{SP^2 . PV} \cdot \frac{PO^2}{PV^2}$$

$$\propto \frac{PO^2}{PV^3} \propto \frac{PG^3}{PM^3} \cdot \frac{1}{PO}:$$

or, if $u$ be the component of the velocity perpendicular to the direction of the force,

$$F = \frac{2u^2}{PO} \cdot \frac{PG^3}{PM^3}.$$

Also velocity at $P = \dfrac{h}{SY} = u \cdot \dfrac{SP}{SY} = u \cdot \dfrac{PG}{PM}$.

*Illustrations.*

1. In the cycloid, force acting parallel to the axis,

$$2PG = PO, \text{ and } PM \propto PG^2;$$

$$\therefore \text{ Force} \propto \frac{PG^2}{PM^3} \propto \frac{1}{PM^2} \propto \frac{1}{PO^4}.$$

$$\text{Velocity} \propto \frac{PG}{PM} \propto \frac{1}{PG} \propto \frac{1}{PO}.$$

2. In the catenary, repulsive force parallel to the direction of gravity, $AM$ being the directrix,

$$PO = PG \propto PM^2; \quad \therefore F \propto PM \propto PO^{\frac{1}{2}}.$$

Also velocity $\propto PM$.

XV.

(?) 1. If a cycloid be described under the action of a force acting in a direction parallel to the base, the force at any point varies inversely as $AM.MQ$; $AM$, $MQ$ being abscissæ and ordinates of the corresponding point of the generating circle.

2. A catenary is described under the action of a horizontal force, prove that the force varies as $n+1$, where $n$ is a number which varies inversely as the subnormal, measured on the directrix.

### Prop. IX. Problem IV.

*If a body revolves in an equiangular spiral, required the law of centripetal force tending to the pole of the spiral.*

Draw $SY$ from $S$ the pole of the spiral perpendicular to the tangent $PY$, and let $PV$ be the chord of curvature at $P$, whose direction passes through $S$; then, since the angle $SPY$ is constant, $SY$ varies as $SP$, also $PV$ varies as $SP$; therefore the centripetal force varies inversely as $SY^2.PV$, and therefore inversely as $SP^3$.

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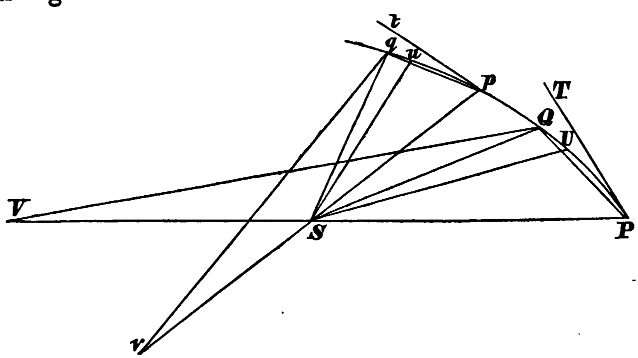
*Notes on the Proposition.*

130. To shew that $PV \propto SP$, and that the arcs subtending equal angles are similar.



Let $PQ$, $pq$ be two arcs of an equiangular spiral which subtend equal angles at $S$, and let $SU$, $Su$ be radii making equal angles with $SP$, $Sp$ respectively,

$$SU : SP :: Su : Sp;$$

$$\therefore\ PQ,\ pq \text{ are similar arcs.}$$

Also, if two circles be described having the same tangents $PT$, $pt$ as the curve at $P$, $p$, and passing respectively through $Q$ and $q$, and $PSV$, $pSv$, be chords of these circles, since the angles of their segments $PQV$, $pqv$, viz. $QPT$, $qpt$ are equal, the segments are similar;

$$\therefore\ PV : pv :: QP : qp :: SP : Sp;$$

$\therefore$ $PV \propto SP$, and $PV$ is ultimately the chord of curvature through $S$ when the angle $PSQ$ is diminished indefinitely.

131. If $F$ be the measure of the accelerating effect of the force tending to the pole of the equiangular spiral, $\angle\, SPY = \alpha$,

$$F = \frac{2h^2}{SY^2\, PV} = \frac{2h^2}{SP^2 \sin^2 \alpha . 2SP} = \frac{h^2}{\sin^2 \alpha . SP^3} = \frac{\mu}{SP^3}.$$

132. *To find the velocity in the equiangular spiral, force tending to the pole.*

$$V^2 = F.\frac{PV}{2} = \frac{\mu}{SP^3}.SP = \frac{\mu}{SP^2};$$

$$\therefore\ V = \frac{\mu^{\frac{1}{2}}}{SP} \propto \frac{1}{SP}.$$

133. *To find the time of describing any arc of the equiangular spiral.*

Let $AL$ be any arc $SA$, $SL$ bounding radii, $P$ the time of describing the arc. Then, as proved in page 30,

$$\text{area } SAL = \tfrac{1}{4}(SA^2 \sim SL^2)\tan\alpha = \tfrac{1}{2}\, h.P;$$

$$\therefore P = \frac{SA^2 \sim SL^2}{2h}\tan\alpha = \frac{SA^2 \sim SL^2}{2\mu^{\frac{1}{2}}\cos\alpha}.$$

*Illustration.*

**In any orbit described under the action of a force tending to any point $S$, when the angle between the tangent $PY$ and the radius $SP$ is a maximum or minimum, the velocity is equal to the velocity in a circle at the same distance about the same force in the center.**

For, the curve near this point may be considered an equiangular spiral ultimately, since the angle is constant for a short time;

$$\therefore \text{ the chord of curvature is } = 2SP,$$

$$\therefore V^2 = F \cdot SP.$$

## XVI.

1. In different equiangular spirals, described under the action of forces tending to the poles which are equal at equal distances, shew that the angular velocity varies at any point directly as the force and the perpendicular on the tangent.

2. The angular velocity of the perpendicular on the tangent is equal to that of radius.

3. The velocity of approach towards the focus, called the paracentric velocity, varies inversely as the distance.

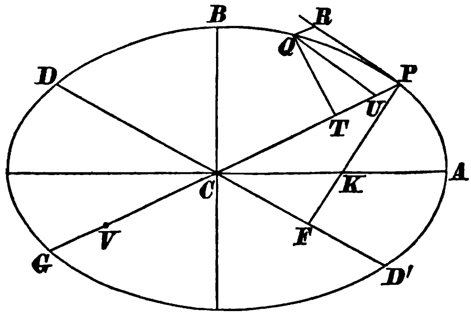
4. Deduce from the time in an equiangular spiral, the time of passing from one point to another, when a body moves along a straight line with a velocity which varies inversely as the distance from a fixed point in that line.

## Prop. X. Problem V.

*If a body is revolving in an ellipse, to find the law of centripetal force tending to the center of the ellipse.*

Let $CA$, $CB$ be the semiaxes of the ellipse, $P$ the position of the body at any time, $PCG$, $DCD'$, conjugate diameters, $Q$ a point near $P$, $QT$, $PF$ perpendiculars from $Q$

and $P$ on $PG$, $DD'$, $QU$ an ordinate to $PCG$, $QR$ a subtense parallel to $CP$.




$$\text{Then } F = \frac{2h^2}{CP^2} \cdot \frac{QR}{QT^2} \text{ ultimately.}$$

But by similar triangles $QTU$, $PFC$,

$$\frac{QT^2}{QU^2} = \frac{PF^2}{CP^2}, \text{ and } \frac{QU^2}{PU.UG} = \frac{CD^2}{CP^2};$$

$$\therefore \frac{QT^2}{PU.UG} = \frac{PF^2.CD^2}{CP^4} = \frac{AC^2.BC^2}{CP^4},$$

and $PU = QR$, and $UG = 2CP$ ultimately;

$$\therefore \frac{QT^2}{2QR} = \frac{AC^2.BC^2}{CP^3} \text{ ultimately;}$$

$$\therefore F = \text{limit of } \frac{2h^2.QR}{CP^2.QT^2} = \frac{h^2.CP}{AC^2.BC^2} \propto CP;$$

therefore the force is proportional to the distance from the center.

*Aliter.*

If $CY$ be perpendicular on the tangent at $P$,

$$F = \frac{2h^2}{CY^2} \frac{QR}{PQ^2} \text{ ultimately,}$$

$$\text{and } \frac{QU^2}{PU.UG} = \frac{CD^2}{CP^2};$$

but $QU = PQ$, and $UG = 2CP$, ultimately, also $PU = QR$;

$$\therefore \frac{PQ^2}{QR} = \frac{2CD^2}{CP}, \quad \text{and } CY = PF;$$

$$\therefore F = \frac{2h^2}{CY^2} \cdot \frac{CP}{2CD^2} = \frac{h^2}{PF^2 \cdot CD^2} \cdot CP$$

$$= \frac{h^2}{AC^2 \cdot BC^2} \cdot CP \propto CP.$$

COR. 1. And conversely, if the force be as the distance, a body will revolve in an ellipse having its center in the center of force, or in a circle, which is a particular kind of ellipse.

COR. 2. And the periodic times will be the same in all ellipses described by bodies about the same center of force.

For the periodic time in any ellipse

$$= \frac{2 \times \text{area of ellipse}}{h} = \frac{2\pi AC \cdot BC}{h},$$

and the forces at different distances in the same or different ellipses vary as the distance; $\therefore \frac{h^2}{AC^2 \cdot BC^2}$ or $\mu$ is the same in different ellipses, therefore the periodic times in different ellipses is the same, and $= \frac{2\pi}{\sqrt{\mu}}$.

## SCHOLIUM.

If the center of an ellipse be supposed at an infinite distance, the ellipse becomes a parabola, and the body will move in this parabola; and the force, now tending to

a center at an infinite distance, will be constant and act in parallel lines. This theorem is due to Galileo. And if the parabola be changed into a hyperbola, by the change of inclination of the plane cutting the cone, the body will move in this hyperbola under the action of a repulsive force tending from the center.

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*Velocity in an Ellipse about the Center.*

134. *To find the velocity in the elliptic orbit under the action of a force tending to the center, the measure of whose accelerating effect is $\mu \times$ distance.*

$$\text{Velocity at } P = \frac{h}{CY} = \frac{h \cdot CD}{CY \cdot CD} = \frac{h \cdot CD}{AC \cdot BC},$$

$$\text{and } \mu = \frac{h^2}{AC^2 \cdot BC^2};$$

$$\therefore \text{ velocity at } P = \sqrt{\mu} \cdot CD.$$

*Aliter.*

$$(\text{Vel.})^2 \text{ at } P = F \cdot \frac{PV}{2} = \mu\, CP \cdot \frac{CD^2}{CP};$$

$$\therefore \text{ vel. at } P = \sqrt{\mu} \cdot CD.$$

135. *To compare the velocity in an ellipse about the center with the velocity in a circle at the same distance.*

$$(\text{Velocity})^2 \text{ in a circle rad. } CP = \mu\, CP \cdot CP;$$

$$\therefore \text{ vel. at } P : \text{vel. at circle (rad. } CP) :: CD : CP.$$

136. *If a hyperbolic orbit be described under the action of a repulsive force tending from the center, the force varies as the distance, and the velocity at any point as the diameter of the conjugate hyperbola parallel to the tangent at the point.*

This may be proved exactly as in the case of the ellipse, employing the proper figure.

137. ***To find the time in any arc of an elliptic orbit about a force tending to the center.***

Let $P$ be any point of the orbit, $Q$ the corresponding point in the auxiliary circle to the ellipse,

$$\text{time from } A \text{ to } P \propto \text{area } ACP \propto \text{area } ACQ \propto \angle\, ACQ,$$

$$\text{and periodic time} = \frac{2\pi}{\sqrt{\mu}};$$

$$\therefore \text{ time in } AP : \frac{2\pi}{\sqrt{\mu}} :: \angle\, ACQ : \text{four right angles};$$

$$\therefore \text{ time in } AP = \text{circular measure of } ACQ \div \sqrt{\mu}.$$

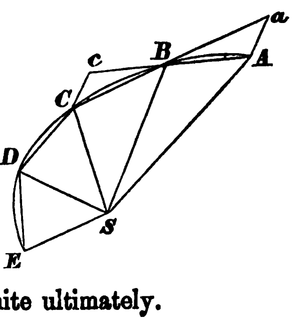
*Notes.*

138. Cor. 1. The proposition asserts only that if a body revolve in an ellipse under the action of a force tending to the center, the force would vary as the distance; but it does not prove that a body *can* move in such an ellipse.

But this can be shewn as a particular case of the following general proposition.

139. Prop. *Let* ABCDE *be any plane curve,* S *any point in the plane, to shew that,* generally, *the curve can be described under the action of a force tending to or from* S, *with finite velocity, the velocity at any given point being any given velocity.*

For arcs $AB$, $BC$, ... can be measured from any point $A$, along the curve, such that the areas $SAB$, $SBC$, ... are all equal, and of any magnitude. Also a body can be made, by some force, to move along the curve with finite velocity, so as to describe the arcs $AB$, $BC$, ... in equal times, unless the tangent to one of the arcs, as $DE$, pass through $S$, in which case, if the arcs be indefinitely diminished, $DE : AB$ is not finite ultimately.



Hence by Prop. II. a body *can* move with finite velocity under the action of some force tending to or from $S$, *generally*.

And since in making the motion of the body such that it shall describe equal areas in equal times we are only concerned with the ratio of the velocities, the velocity at any point $A$ may be any given velocity. Q. E. D.

140. Cor. 1. Or if we please we may suppose the force at any point any given force; for the force depends only upon the velocity and the form of the curve, and the form of the curve being known at $A$, the velocity may be chosen so as to make the force equal to the given force.

141. Cor. 2. The ratio of the velocities is the same at two given points, for all forces tending to a given center, under the action of which the curve can be described.

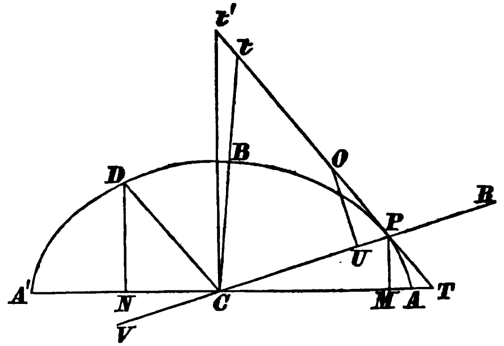
142. Cor. 3. The proposition is true with respect to a point $S$ moving uniformly in a straight line.

143. Cor. 4. Hence a body can move *throughout* any ellipse under the action of a centripetal force tending to the center or focus, or any point within the ellipse, with a finite velocity, since no point within an ellipse lies on any tangent.

144. Cor. 5. In the case of a circle, $S$ being an external point, a body can move with finite velocity under the action of a force tending to the point $S$, in the portion which is concave to $S$, *or* from $S$, in that which is convex to $S$; but not from one portion to the other.

145. *If at a given point the velocity of a body be known, and the direction of its motion; to determine the curve which the body will describe under the action of a given centripetal force, which varies as the distance from the point to which it tends.*

Let $Pt$ be the direction of motion at $P$, $V$ the velocity at $P$, $\mu \,.\, CP$ the measure of the accelerating effect of the force tending to $C$.




Let $PV$ in $PC$ produced be four times the space through which a body must move from rest under the action of the force at $P$ continued constant, in order to acquire the given velocity $V$; when $V^2 = 2\mu\, CP \,.\, \frac{1}{4} PV$.

Draw $CD$ parallel to $Pt$, a mean proportional to $CP$ and $\frac{1}{2}PV$, and let an ellipse be constructed with $CP$, $CD$ as semi-conjugate diameters, then $PV$ is the chord of curvature at $P$ through $C$.

In this ellipse let a body revolving under the action of the given force arrive at the point $P$, at this point it will be moving in the direction $Pt$, also (vel.)$^2$ at $P = \mu CD^2 = \mu \,.\, CP \,.\, \frac{1}{2}PV = V^2$, or the velocity at $P$, in the constructed ellipse, is $V$. Hence the body revolving in this ellipse is under the same circumstances as the proposed body, in all respects which can influence the motion of a body; therefore the proposed body will describe the ellipse constructed as above.

*Lemmas in Conic Sections.*

1. If $TPt$ be a tangent at $P$ to an ellipse, $CD$ the parallel semidiameter $CT$, $Ct$ in the direction of *any* conjugate diameters, $PT \,.\, Pt = CD^2$.

$$\text{For } PT : MT :: CD : CN,$$

$$Pt : CM :: CD : CN;$$

$$\therefore PT.Pt : CM.MT :: CD^2 : CN^2;$$

and it is easily shewn that $CN^2 + CM^2 = CA^2$;

since $CP$ and $CD$ are conjugate;

$$\therefore CN^2 = CM.CT - CM^2 = CM.MT;$$

$$\therefore PT.Pt = CD^2.$$

2. The converse is also true, viz. if $PT.Pt' = CD^2$ be given, $CT$, $Ct'$ are in direction of conjugate diameters. Draw $Ct$ conjugate to $CT$;

$$\therefore PT.Pt = CD^2;$$

$$\therefore Pt = Pt',$$

or $Ct'$ and $CT$ are conjugate.

146. *Geometrical construction for the position and magnitude of the axes of the elliptic orbit, described by a body about the center, when the velocity at a given point is known, and the direction of motion.*

Produce $CP$ to $R$, making $PR$ a third proportional to $CP$ and $CD$; bisect $CR$ in $U$, and draw $UO$ perpendicular to $CR$, meeting $tP$ in $O$, and with center $O$ describe a circle passing through $C$, $R$, and cutting $tPT$ in $t$ and $T$;

$$\therefore PT.Pt = CP.PR = CD^2;$$

therefore $CT$, $Ct$ are in direction of conjugate diameters, and $TCt$ is a right angle; therefore $CT$, $Ct$ are the directions of the axes of the ellipse, and if $PM$, $Pm$ be perpendicular to these diameters, the semiaxes are respectively mean proportionals between $CM$, $CT$ and $Cm$, $Ct$. Q. E. F.

147. *Equations for determining the position and dimensions of the orbit.*

Let $\mu . R$ be the measure of the accelerating effect of the force at distance $R$, $V$ the velocity, $a$ the angle between the direction of motion at the given point $P$ and $CP$, $CP = R$, $a$, $b$ the semiaxes of the ellipse, $\varpi$ the angle which the larger axis makes with the distance $CP$.

$$\text{Then } V^2 = \mu . CD^2,$$

$$\text{and } CD^2 + CP^2 = a^2 + b^2;$$

$$\therefore\ a^2 + b^2 = \frac{V^2}{\mu} + R^2 \ \ldots\ldots\ldots\ldots\ldots (1)$$

$$\text{Also } h = V . R \sin a = \sqrt{\mu} . ab;$$

$$\therefore\ ab = \frac{V . R \sin a}{\sqrt{\mu}} \ \ldots\ldots\ldots\ldots\ldots (2)$$

and by the properties of the ellipse,

$$\frac{R^2}{a^2} \cos^2 \varpi + \frac{R^2}{b^2} \sin^2 \varpi = 1 \ \ldots\ldots\ldots\ldots\ldots (3)$$

The equations (1) (2) and (3) determine $a$, $b$ and $\varpi$, whence the position and magnitude of the ellipse is determined.

Or, instead of (3) we can obtain $\varpi$ without previously determining $a$ and $b$.

For let $PM$, $DN$ be perpendicular to the major axis.

$$\text{Then } V \cos (a - \varpi) = \sqrt{\mu}\, CD \cos DCN = \sqrt{\mu}\, CN,$$

$$V \sin (a - \varpi) = \sqrt{\mu}\, CD \sin DCN = \sqrt{\mu}\, DN;$$

$$\therefore\ V^2 \sin (a - \varpi) \cos (a - \varpi) = \mu CN . DN = \mu\, CM . PM$$

$$= \mu R^2 \cos \varpi \sin \varpi;$$

$$\therefore\ V^2 \sin 2(a - \varpi) = \mu R^2 \sin 2\varpi;$$

$$\therefore\ V^2 (\sin 2a \cot 2\varpi - \cos 2a) = \mu R^2;$$

$$\therefore\ \cot 2\varpi = \frac{\mu R^2}{V^2} \operatorname{cosec} 2a + \cot 2a \ldots\ldots\ldots (4)$$

whence $\varpi$ is known immediately from the initial circumstances of the motion.

148. If the force be repulsive, the equations for determining $a$, $b$, $\varpi$ are

$$a^2 - b^2 = R^2 - \frac{V^2}{\mu} \dots\dots\dots\dots\dots\dots (1)$$

$$ab = \frac{VR \sin a}{\sqrt{\mu}} \dots\dots\dots\dots\dots\dots (2)$$

$$\text{and } \frac{R^2}{a^2} \cos^2 \varpi - \frac{R^2}{b^2} \sin^2 \varpi = 1 \dots\dots\dots\dots (3)$$

The direction and magnitude of the axes of the hyperbola may be determined geometrically with ease, by observing that the asymptotes are the diagonals of the parallelograms of which the conjugate diameters are sides.

### *Apses.*

149. In any orbit described under the action of a force tending to a fixed center, a point at which the direction of the motion is perpendicular to the central distance is called an *apse*, the distance is called an *apsidal distance*, and the angle between consecutive apsidal distances is called an *apsidal angle*.

Thus, in the ellipse about the center, the extremities of the axes are four apses, and there are two different apsidal distances, and every apsidal angle is a right angle.

In the circle about an internal point the apses are at the greatest and least distances, and the apsidal angle is two right angles.

150. *In a central orbit described under the action of forces tending to a fixed point, each apsidal distance divides the orbit symmetrically.*

It is easily shewn that in any orbit described under the action of a force tending to the center, if another body be projected at any point with the same velocity in the opposite direc-

tion it will proceed to describe the same orbit in the reverse direction, under the action of the same force.

For, let $ABC$, fig. Art. 139, be a portion of the polygon described, whose limit is the path of the body, produce $AB$ to $c$, and $CB$ to $a$, making $Bc = AB$, and $Ba = CB$.

The impulse at $B$ is measured by $cC$ when the body describes $ABC$, and if the motion be reversed, the same impulse at $B$ would cause the body to move in $BA$, since $aA = cC$. And the same is true throughout the polygonal path, hence the assertion is true for the polygonal path under the action of impulses which are always the same at the same points, and therefore it is true in the limit as stated for the curvilinear motion.

Hence the path of the body from an apse being similar and equal to the path which would be described if the motion were reversed at the apse, is similar to the path described in approaching the apse; whence the proposition is established.

151. *There are only two different apsidal distances, and all apsidal angles are equal.*

For, after passing a second apse, the curve being symmetrical on both sides, a third apse will be in such a position that the apsidal distance is the same as for the first apse, and all the apsidal angles are shewn similarly to be equal.

152. Cor. Hence a central orbit can never re-enter unless the ratio of the apsidal angle to a right angle be commensurable, and if it be so the curve will always re-enter.

*Illustrations.*

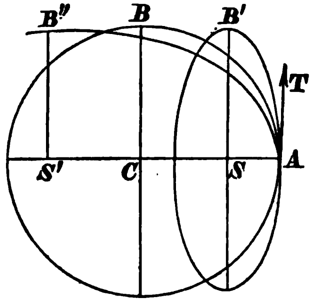
1. A body revolves in a circular orbit about a force which varies as the distance, and tends to the center of the circle; the center of force is suddenly transferred to a point in the radius, which at the moment of change passes through the body: to find the subsequent motion of the body.

(1) Since the force varies as the distance and is attractive, the orbit is an ellipse.

(2) And, since the force is a finite force, the body will move in the same direction as before, at the moment of the change.

(3) Also, the velocity will for the same reason be unaltered, at that moment, since the force requires a finite time to produce an effect.

Let $CA$ be the radius passing through the body at the moment of change, $\mu . CA$ the force at distance $CA$, $V$ the velocity in the circle.




Then $V^2 = \mu CA . CA = \mu CA^2$; and, if $S$ be the new point to which the force tends, let $AB'$ be the ellipse described (1); $SA$ is one of the semiaxes of the ellipse since $A$ is an apse (2), and $SB'$ being the other, if a body revolved in this ellipse round $S$, $\mu SB'^2$ would be the (vel.)$^2$ at $A$, the same as in the circle (3); or $\mu . SB'^2 = \mu CA^2$, and $\therefore SB' = CA = CB$;

$\therefore$ the two semiaxes, $SA$ and $CA$ are determined, and their position is known, and $\therefore$ the ellipse completely determined.

The ellipse lies without the circle at $A$, because, the velocity being unaltered, the force has been diminished in the ratio of $CA : SA$, and therefore the curvature diminished in that ratio.

If $S$ had been in $AC$ produced, the force would have been increased, and the orbit within the circle near $A$.

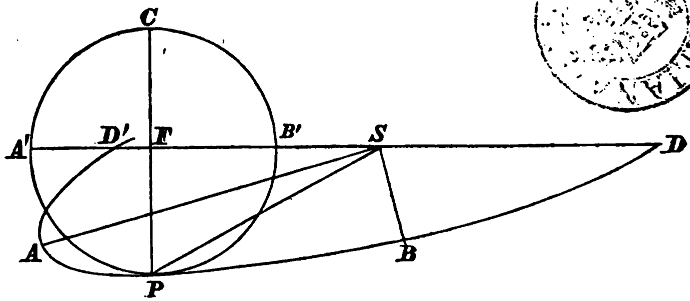
The greatest distance from $CA$ which the body reaches is in all cases the same for this law of force, because the component of the force perpendicular to $CA$ is the same at the same distance from $CA$ in whatever curve the body moves; therefore, the velocity being the same at $A$, the velocity perpendicular to $AC$ is destroyed by the force at the same distance from $AC$, in each orbit.

2. A body is describing a circle about a force which varies as the distance and tends to the center; if the center to which the force tends be suddenly transferred to a point in the circum-

ference $60^\circ$ from the position of the particle, to determine the orbit described.

The orbit is an ellipse, since the force is attractive.






Let $P$ be the position of the body at the instant the center of force is transferred from $C$, the center of the circle, to $S$, then $SCP$ is an equilateral triangle.

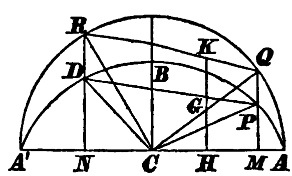
Draw $DSD'$ perpendicular to $CP$, meeting it in $F$, and take $SD = SD' = SP$. Construct an ellipse having $SP$, $SD$ as semiconjugate diameters; $SA$, $SB$ the semiaxes bisect the angles $PSD$, $PSD'$; the velocity at $P$ is $\sqrt{\mu}\,.\,SP = \sqrt{\mu}\,.\,SD$; $\therefore$ this ellipse is the subsequent orbit whose major-axis bisects the angle $PSF$.

Prove the following construction:

On $CP$ as diameter describe a circle cutting $SD'$ in $B'$, $A'$; $SA'$, $SB'$ are the lengths of the semiaxes.

Explain why the orbit is exterior to the circle.

3. Two bodies whose masses are $m$, $m'$ revolve in an ellipse, under the action of a force tending to the center; shew that if they are at one time at the extremities of two conjugate diameters, they will always be so, and in this case find the locus of their center of gravity.




Let $P$, $D$ be their positions at any time, $CP$, $CD$ being semiconjugate diameters. Let the ordinate $MPQ$, $NDR$ meet the auxiliary circle in $Q$ and $R$.

Since the angles $ACQ$, $ACR$ are

always proportional to the times, $\therefore RCQ$ will always be a right angle; $\therefore$ the bodies will always be at the extremities of conjugate diameters.

Let $GH$ be the ordinate of their center of gravity.

Join $RQ$ and produce $HG$ to $RQ$ in $K$;

$$\therefore KH : GH :: QM : PM \text{ a constant ratio,}$$

$$\text{and } RK : KQ :: DG : GP \text{ ..................;}$$

$$\therefore CK \text{ is constant, or the locus of } K \text{ is a circle,}$$

$\therefore$ the locus of $G$ is an ellipse whose axes are proportional to those of $APD$.

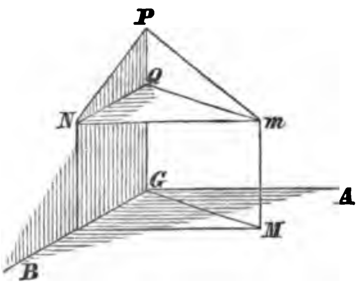
Shew that the semimajor-axis : $CA$ :: $m^2 + m'^2 : (m + m')^2$.

4. A body is composed of matter which attracts with a force varying as the distance; shew that however a particle be projected, unless it strikes the body, it will describe its orbit in the same periodic time.

This is obvious immediately from the following proposition relating to attractive forces.

PROP. *A body composed of matter which attracts with a force varying as the distance attracts any particle in the same manner as if the body were collected in the center of gravity.*

Let $G$ be the center of gravity of the body, $P$ the attracted particle, $m$ a particle of the body of mass $m$, and let $AGB$ be a plane through $G$ perpendicular to $PG$; $PGA$, $PGB$ two planes containing $P$ and perpendicular to each other.




Draw $mM$, $mN$ perpendicular to $AGB$, $PGB$, and join $MG$, draw $mQ$ perpendicular to $PG$, and $\therefore$ parallel to $MG$.

The attraction of $m$ on $P$ is measured by $m \cdot mP$ and its component perpendicular to $AGB$ by

$$m \cdot PQ = m \cdot (PG - GQ);$$

$\therefore$ the attraction of the body on $P$ in the direction perpendicular to $AGB$ is measured by

$$\Sigma\{m(PG - GQ)\} = \Sigma(m) . PG - \Sigma(m . mM) = \Sigma(m) . PG,$$

since $\Sigma(m . mM) = 0$, the center of gravity being in $AGB$.

The attraction perpendicular to $PGB$ is measured by

$$\Sigma(m . mN) = 0,$$

and similarly for $PGA$; $\therefore$ $\Sigma(m) PG$ measures the whole attraction, whence the truth of the proposition.

5. A body moves in an ellipse under the action of a force varying as the distance: if the velocity at any point be slightly increased by $\frac{1}{n}$th of itself, find the consequent changes in the axes of the ellipse.

If the body be at the end of one of the equal conjugate diameters when the change takes place, shew that each axis is increased by $\frac{1}{2n}$th of itself, and that the apse line regredes through a small angle whose circular measure is $\frac{1}{n}\frac{ab}{a^2 - b^2}$.

When $V$ is changed to $V\left(1 + \frac{1}{n}\right)$, let the corresponding changes of $a$, $b$, and $\varpi$ be $a\alpha$, $b\beta$, and $\gamma$: $\alpha$, $\beta$, $\gamma$, and $\frac{1}{n}$ being so small that we may neglect their squares.

Then, by the equations of Art. 147, and notes (1), (2), (3) in illustration 1,

$$a^2(1 + \alpha)^2 + b^2(1 + \beta)^2 = \frac{V^2}{\mu}\left(1 + \frac{1}{n}\right)^2 + R^2,$$

$$\text{and } a^2 + b^2 = \frac{V^2}{\mu} + R^2;$$

$$\therefore a^2\alpha + b^2\beta = \frac{V^2}{\mu n} \text{ .....................(1).}$$

$$a^2 b^2 (1 + \alpha)^2 (1 + \beta)^2 = \frac{V^2 R^2 \sin^2 a}{\mu}\left(1 + \frac{1}{n}\right)^2,$$

$$\text{and } a^2 b^2 \qquad = \frac{V^2 R^2 \sin^2 a}{\mu};$$

$$\therefore\ \alpha + \beta = \frac{1}{n} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

and $\alpha$ and $\beta$ are known from (1) and (2).

In the particular case proposed,

$$V^2 = \mu \,.\, CD^2 = \tfrac{1}{2}\mu\,(a^2 + b^2), \text{ and } R^2 = \frac{a^2 + b^2}{2},$$

$$a^2\alpha + b^2\beta = \frac{1}{2n}(a^2 + b^2);$$

$$\therefore\ (a^2 - b^2)\,\alpha = \frac{1}{2n}(a^2 - b^2);$$

$$\therefore\ \alpha = \frac{1}{2n} = \beta.$$

$$\text{Also } \frac{R^2}{a^2}\cos^2(\varpi + \gamma) + \frac{R^2}{b^2}\sin^2(\varpi + \gamma) = 1 + \frac{1}{n},$$

$$\frac{R^2}{a^2}\cos^2\varpi + \frac{R^2}{b^2}\sin^2\varpi = 1;$$

$$\therefore\ \left(\frac{R^2}{b^2} - \frac{R^2}{a^2}\right)\{\sin^2(\varpi+\gamma) - \sin^2\varpi\} = \frac{1}{n};$$

$$\therefore\ \left(\frac{a^2 + b^2}{2b^2} - \frac{a^2 + b^2}{2a^2}\right)\sin(2\varpi + \gamma)\sin\gamma = \frac{1}{n};$$

$\therefore\ \gamma$, being expressed in circular measure, $= \dfrac{2a^2 b^2}{n(a^4 - b^4)\sin 2\varpi}$;

and since the axes bisect the angles between equal conjugate diameters,

$$PCD = \pi - 2\varpi,$$

$$ab = CP \,.\, CD \sin 2\pi = \tfrac{1}{2}(a^2 + b^2)\sin 2\varpi;$$

$$\therefore\ \gamma = \frac{ab}{n(a^2 - b^2)}.$$

## XVII.

1. Shew that the velocity in an ellipse about the center is the same as that in a circle at the same distance at the points whose conjugate diameters are equal.

2. A body is revolving in a circle under the action of a force tending to the center, the law of force at different distances being that the force varies as the distance; find the orbits described when the circumstances are changed at any point as follows:

(1) If the force be increased in the ratio of $1 : n$.

(2) If the velocity be increased in the ratio $1 : n$.

(3) If the force become repulsive and of the same magnitude.

(4) If the direction be changed by an impulse in the direction of the center, measured by the velocity which is equal to that in the circle.

3. A particle is revolving in a circle round a force which varies as the distance; the center of force is suddenly transferred to the opposite extremity of the diameter through the particle and becomes repulsive; shew that the eccentricity of the hyperbolic orbit $= \frac{1}{3}\sqrt{5}$.

4. If a body be projected from an apse, with a velocity double of that in a circle at the same distance, find the position and magnitude of the axes of its orbit.

5. An elastic ball moving in an ellipse about the center, on arriving at the extremity of the minor-axis strikes another ball at rest directly; find the orbits described by both bodies.

6. The particles of which a rectangular parallelopiped is composed attract with a force which varies as the distance, and a body is projected so as to describe a curve on one of the faces supposed smooth; find the periodic time.

7. A body is projected at an angle $\cos^{-1}\dfrac{1}{\sqrt{3}}$ with the distance from a point to which the force tends, varying as the distance from it, and the velocity $= \sqrt{\frac{3}{2}} \times$ velocity in the circle at

the same distance; prove that one axis is double of the other and that the inclination of the major-axis to the distance is $\frac{1}{2}\cos^{-1}\frac{1}{3}$.

8. $CX$, $CY$ are straight lines perpendicular to one another, and a force tends to $C$, and varies as the distance from $C$. If from various points in $CY$ different particles are projected parallel to $CX$ at the same moment, and with the same velocity, they will all arrive at $CX$ at the same time and place.

9. Prove the same, if $CX$, $CY$ be inclined at any angle, and also if at any time the force cease to act.

10. A particle is describing an ellipse about a force tending to the center $C$, and when it arrives at $B$, the extremity of the minor-axis, the force is suddenly transferred to the focus $S$; shew that the major-axis of the new orbit bisects the angle $BSC$. Shew also that if $a$, $b$, $\alpha$, $\beta$ be the semiaxes of the old and new orbits,

$$a - b : a + b :: (\alpha - \beta)^2 : (\alpha + \beta)^2.$$

11. From points in a line $CA$ between $C$ and $A$ particles are projected at right angles to $CA$, with velocities proportional to their distances from $A$, $C$ being a center to which the force tends, and the force varying as the distance; find the ellipse of greatest area which is described.

12. A particle is projected from a point $P$ in a given ellipse to the center $C$ of which a force, varying as the distance, tends: the direction of projection is perpendicular to the major-axis, the velocity is that in a circle whose radius is $CS$; prove that the major-axis of the orbit is that of the given ellipse, and that $CP^2 =$ the sum of the squares of the semi minor-axes of the orbit and given ellipse.

$$\text{Also } \tan^2\varpi = \left(1 - \frac{c^2}{a^2}\right)\left(\frac{c^2}{b^2} - 1\right).$$

13. A number of particles move in hyperbolas, under the action of the same repulsive force from their common center.

Shew that, if the transverse axes coincide, and the particles start from the vertex at the same instant, they will always lie in a straight line perpendicular to the major-axis.

14. If the hyperbolas in the last problem all have the same asymptotes, and if the particles start from their vertices at the same instant, shew that they will always lie in a straight line through the center.

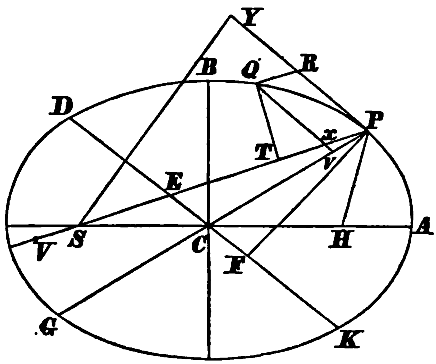
15. Four equal bodies are placed in a smooth elliptic groove at the extremities of equal conjugate diameters, and are acted on by their mutual attraction, which varies as the distance. Shew that if they be projected with the same velocity equal to that with which they would revolve in a circle, passing through them all, they would exert no pressure on the groove, the sum of the squares of their velocities would never vary.

## SECTION III.

### *On the Motion of Bodies in Conic Sections, under the action of Forces tending to a Focus.*

#### Prop. XI. Problem VI.

*If a body is revolving in an ellipse, to find the law of force tending to a focus of the ellipse.*




Let $S$ be the focus to which the force tends, $P$ the position of the body at any time, $PCG$, $DCK$ conjugate diameters, $Q$ a point near $P$, $QT$, $PF$ perpendiculars from $Q$, $P$ on $SP$, $DCK$, $PR$ a tangent at $P$, $QR$ parallel to $SP$, $Qxv$ to $PR$ meeting $SP$ in $x$ and $PC$ in $v$, and let $SP$, $DCK$ intersect in $E$. Then $PE = AC$.

Then $F = \dfrac{2h^2}{SP^2} \cdot \dfrac{QR}{QT^2}$, ultimately, when $QP$ is indefinitely diminished.

But by similar triangles, $QTx$, $PFE$,

$$\frac{QT^2}{Qx^2} = \frac{PF^2}{PE^2} = \frac{PF^2}{AC^2} = \frac{BC^2}{CD^2}.$$

Now, $\frac{Qv^2}{Pv \cdot vG} = \frac{CD^2}{CP^2}$ by the properties of the ellipse,

and $\frac{Pv}{QR} = \frac{Pv}{Px} = \frac{CP}{PE}$ by similar triangles;

$$\therefore \frac{Qv^2}{QR \cdot vG} = \frac{CD^2}{CP \cdot AC},$$

and $vG = 2CP$, $Qx = Qv$, ultimately;

$$\therefore \frac{QT^2}{QR} = \frac{2BC^2}{AC} = L, \text{ ultimately,}$$

if $L$ be the latus rectum of the ellipse.

$$\therefore F = \frac{2h^2}{L} \cdot \frac{1}{SP^2} \propto \frac{1}{SP^2}.$$

*Aliter.*

Since the force tending to the center of an ellipse, under the action of which the ellipse can be described, varies directly as the distance $CP$ from the center $C$; let $CE$ be drawn parallel to the tangent $PR$ to the ellipse; then if $S$ be any point within the ellipse and $SP$, $CE$ intersect in $E$, force tending to $C$ : force tending to $S$

$$:: CP \cdot SP^2 : PE^3 \text{ (Prop. VII. Cor. 3);}$$

$$\therefore \text{force tending to } S \propto \frac{PE^3}{SP^2},$$

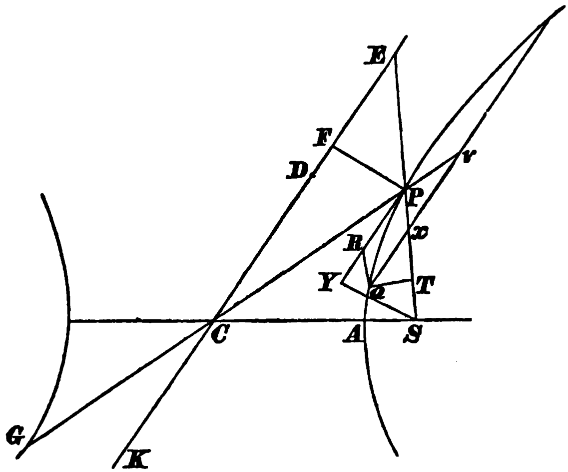
if $S$ be the focus of the ellipse, $PE = AC$ is constant;

$$\therefore \text{force tending to } S \propto \frac{1}{SP^2}.$$

## Prop. XII. Problem VII.

*If a body is revolving in an hyperbola, to find the law of force tending to a focus of the figure.*

The investigation is exactly the same as in the last proposition, employing the subjoined figure.



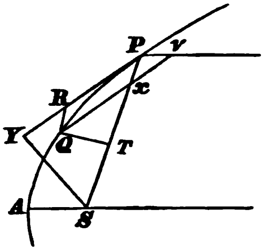

Also, *repulsive* force from $C \propto CP$, and by Prop. VII. Cor. 3, force from $C$ : force to $S$ :: $CP \,.\, SP^2 : PE^3$, whence force to $S \propto \dfrac{1}{SP^2}$, since $PE$ is constant.

In the same manner as in these propositions it can be shewn that the repulsive force tending from a focus, under the action of which the body describes the opposite branch of the hyperbola, varies inversely as the square of the distance.

## Prop. XIII. Problem VIII.

*If a body is moving in a parabola, to find the law of force tending to the focus.*

Let $S$ be the focus of the parabola, $P$ the position of the body at any time, $Q$ a point near $P$, $PRY$ a tangent at $P$, $QR$ parallel to $SP$, $Qxv$ to $PR$, meeting $SP$ in $x$ and the diameter through $P$ in $v$, $QT$, $SY$ perpendicular to $SP$, $PY$ respectively.




Then $F = \frac{2h^2}{SP^2} \cdot \frac{QR}{QT^2}$, ultimately,

when $QP$ is indefinitely diminished.

Since $SP$, $Pv$ make equal angles with the tangent, $Pxv$ is an isosceles triangle, therefore $Pv = Px = QR$, and by similar triangles

$$\frac{QT^2}{Qx^2} = \frac{SY^2}{SP^2} = \frac{AS \cdot SP}{SP^2} = \frac{AS}{SP},$$

$$\text{and } Qv^2 = 4SP \cdot Pv = 4SP \cdot QR;$$

$$\therefore, \text{ since } Qx = Qv, \text{ ultimately,}$$

$$\frac{QT^2}{4SP \cdot QR} = \frac{AS}{SP}, \text{ ultimately, or } \frac{QT^2}{QR} = 4AS = L;$$

$$\therefore\ F = \frac{2h^2}{L} \cdot \frac{1}{SP^2} \propto \frac{1}{SP^2}.$$

Cor. 1. It follows from the last three propositions, that if any body move from the point $P$ in any direction $PR$ with any velocity, and be at the same time acted on by a centripetal force, which is inversely proportional to the square of the distance, the body will move in some one of the conic sections, having a focus in the center of force, and conversely.

For when the focus is given, and the point of contact, and the position of the tangent, a conic section can be de-

scribed which will have a given curvature at that point. But when the force is given and the velocity of the body, the curvature is known; and two orbits touching one another cannot be described with the same centripetal force, and the same velocity at the point of contact.

COR. 2. If the velocity with which a body leaves its position $P$, is such that it could describe the small space $PR$ in some very small time, and in the same time the centripetal force were able to move the same body through the space $RQ$, this body will move in some conic section whose latus rectum is the limit of $\frac{QT^2}{QR}$ when the lines $PR$, $QR$ are indefinitely diminished.

In these corollaries the circle is included as a particular case of an ellipse; and the case is excepted in which the body moves in a straight line to the center of force.

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*Notes.*

153. If $\mu$ be the absolute force, in any conic section about the focus, whose latus rectum is $L$, $\mu = \frac{2h^2}{L}$, and $\mu$ is given when the force at any point is given, or when the velocity at any point in a given conic section is given; for in that case $L$ and $V.SY$ or $h$ are given.

154. If we assume the chord of curvature at any point in an ellipse or hyperbola, we obtain the law of force from the expression $F = \frac{2h^2}{SY^2.PV}$.

$$\text{For, } PV.AC = 2CD^2 = 2SP.HP;$$

$$\text{and } SY^2 : BC^2 :: SP : HP;$$

$$:: SP^2 : \tfrac{1}{2}PV.AC;$$

$$\therefore\ F = \frac{h^2 AC}{BC^2 \,.\, SP^2}.$$

Similarly for the parabola, when

$$PV = 4SP, \text{ and } SY^2 = AS \,.\, SP,$$

$$F = \frac{2h^2}{AS \,.\, SP \,.\, PV} = \frac{h^2}{2AS \,.\, SP^2}.$$

155. Cor. 1. It is assumed in this corollary that a conic section can be described under the action of a force tending to the focus: see Art. 139.

## Prop. XIV. Theorem VI.

*If any number of bodies revolve about a common center, and the centripetal force varies inversely as the square of the distance; the latera recta of the orbits described are in the duplicate ratio of the areas, which the bodies describe in the same time by radii drawn to the center of force.*

For in each orbit the latus rectum is equal to the limit of $\frac{QT^2}{QR}$ (by Cor. 2, Prop. xiii.) when the arc $PQ$ is made indefinitely small.

But $QR$ in a given time is ultimately in the different orbits as the centripetal force, that is, reciprocally as the square of the distance $SP$.

Hence, ultimately, $\frac{QT^2}{QR} \propto QT^2 \,.\, SP^2$, or the latus rectum is in the duplicate ratio of $QT \,.\, SP$ or of twice the area $PSQ$ described in a given small time, and, since the area in each orbit is proportional to the time, $L$ varies as the area described in any given time.

Cor. Hence the whole area of the ellipse, and the rectangle under the axes, which is proportional to it, varies

in a ratio compounded of the subduplicate ratio of the latus rectum and the ratio of the periodic time.

For the whole area is as $QT \times SP$ described in a given time, multiplied by the periodic time.

---

156. We may prove the proposition as follows.

Let $h$, $h'$ be the double areas described in the same time in *any* two of the orbits; $L$, $L'$ the latera recta, then since the centripetal forces vary in the different orbits inversely as the square of the distances,

$$\frac{2h^2}{L\,.\,SP^2} : \frac{2h'^2}{L'\,.\,SP'^2} :: SP'^2 : SP^2,$$

$$\therefore \frac{2h^2}{L} = \frac{2h'^2}{L'};$$

$$\therefore L : L' :: h^2 : h'^2;$$

or the latera recta are in the duplicate ratio of the areas described in a given time.

157. And similarly for the corollary.

Let $P$, $P'$ be the periodic times. Then the areas are as

$$hP : h'P';$$

$$:: L^{\frac{1}{2}}\,.\,P : L'^{\frac{1}{2}}\,.\,P.$$

## Prop. XV. Theorem VII.

*On the same supposition, the squares of the periodic times in ellipses are proportional to the cubes of the major-axes.*

If $P$ be the periodic time in any ellipse, $P\,.\,h \propto$ area of ellipse $\propto AC\,.\,BC$; and, since the force is the same in the different orbits, $BC^2 \propto h^2\,.\,AC$;

$$\therefore P^2 \propto AC^2\,.\,\frac{BC^2}{h^2} \propto AC^3. \qquad \text{Q. E. D.}$$

**Cor. Hence the periodic times in ellipses are the same as in circles whose diameters are equal to the major-axes of the ellipses.**

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158. The periodic time $P$ may be found in terms of the absolute force $\mu$.

For $h.P$ = twice the area of the ellipse

$$= 2\pi AC.BC;$$

and $$\mu = \frac{AC.h^2}{BC^2};$$

$$\therefore\ P = 2\pi AC.\frac{BC}{h} = 2\pi AC.\left(\frac{AC}{\mu}\right)^{\frac{1}{2}}$$

$$= \frac{2\pi AC^{\frac{3}{2}}}{\mu^{\frac{1}{2}}}.$$

159. The time in any arc of an ellipse may be found from the area of the sector $ASP$, and shewn to be

$$\frac{1}{\mu^{\frac{1}{2}}.AC^{\frac{1}{2}}}.(SC.QM + AC.\text{ arc } AQ).$$ See page 35.

*Kepler's Laws.*

160. The three laws known by the name of Kepler's Laws are,

I. That planets move in ellipses having the Sun's center in one focus.

II. That the areas swept out by radii drawn from the planet to the Sun's center are, in the same orbit, proportional to the time of describing them.

III. That the squares of the periodic times are proportional to the cubes of the major-axes.

These laws were discovered by Kepler from observations made on the planet Mars, and stated by analogy as general

laws, which, although not rigidly true as stated by Kepler, are sufficiently near to the truth to have led to the discovery of the law of attraction of the bodies of the solar system.

161. The deviation from complete accuracy is due to the facts that the planets are not of inappreciable mass, that they disturb each other's orbits about the Sun, and by their action on the Sun itself cause the periodic time of each to be shorter than if the Sun were a fixed body, in the subduplicate ratio of the mass of the Sun to the sum of the masses of the Sun and Planet; this error is appreciable although very small, since the largest of the planets, Jupiter, is less than $\frac{1}{1000}$th of the Sun's mass.

*Deductions from Kepler's Laws.*

162. From the law of the equable description of areas, stated as the *second law*, it is deduced, by Prop. II., that the forces acting on the planets are centripetal forces tending to the Sun's center.

But this law gives no information regarding the nature or intensity of the forces.

163. From the elliptic motion of the planets, as asserted in the *first law*, it is deduced, by Prop. XI., that the force which acts upon *each* planet varies inversely as the square of the distance from the center of the Sun.

164. From the relation between the periodic times and lengths of the major-axes, stated in the *third law*, it is inferred, by Prop. XV., that the planets are acted on by the same centripetal force; and that the attraction, being the same for all bodies, independently of their form and substance, is not of the nature of the elective action of chemical or magnetic forces.

165. The same laws hold for the motion of the satellites of Jupiter, Saturn, and Uranus, and the first two for our Moon,

their respective primaries taking the place of the Sun in the statement of the laws.

Hence it is inferred that forces tend to the centers of the planets, varying according to the same law as the forces tending to the Sun.

166. By such deductions the law of gravitation is rendered probable, that *every particle attracts every other particle with a force which varies inversely as the square of the distance.*

The law thus suggested is assumed to be *universally* true, and calculations are made of the effects of the action of the bodies of the solar system upon one another in disturbing their elliptic motion; and also of the disturbances of the motion of the satellites due to the want of exact sphericity in the primaries.

Predictions of the return of comets have been fulfilled, founded on the supposition of the truth of the law, and the existence and position of a planet have been recognized before its discovery, from its assumed action according to this law upon another planet.

Thus the law of gravitation has satisfied every test which could be applied to it, and it is therefore proved to be true as far as our system is concerned.

### Prop. XVI. Theorem VIII.

*On the same supposition, the velocities of the bodies are in the ratio compounded of the inverse ratio of the perpendiculars from the focus on the tangent and the subduplicate ratio of the latera recta.*

For in any two orbits

$$V : V' :: \frac{h}{SY} : \frac{h'}{SY'}$$

$$:: \frac{L^{\frac{1}{2}}}{SY} : \frac{L'^{\frac{1}{2}}}{SY'}.$$

COR. 1. The latera recta of the orbits are in the ratio compounded of the duplicate ratio of the perpendiculars and the duplicate ratio of the velocities.

For $L : L' :: h^2 : h'^2$

$$:: V^2 . SY^2 : V'^2 . SY'^2.$$

COR. 2. The velocities of the bodies, at their greatest and least distances from their common focus, are in the ratio compounded of the ratio of the distances inversely, and the subduplicate ratio of latera recta directly.

For the perpendiculars on the tangents are these very distances.

COR. 3. And therefore the velocity in a conic section, at the greatest or least distance from the focus, is to the velocity in a circle at the same distance from the center in the subduplicate ratio of the latus rectum to twice that distance.

For the latus rectum of a circle is the diameter, therefore if $SA$ be the greatest or least distance, velocity in the conic section : velocity in the circle

$$:: \frac{L^{\frac{1}{2}}}{SA} : \frac{(2SA)^{\frac{1}{2}}}{SA} :: L^{\frac{1}{2}} : (2SA)^{\frac{1}{2}}.$$

COR. 4. The velocities of bodies revolving in ellipses are, at their mean distances from the common focus, the same as the velocities of bodies revolving in circles at the same distances; that is, (by Cor. 6, Prop. IV.) in the inverse subduplicate ratio of the distances.

For the perpendiculars are now the semiaxes minor, or $SY = BC$ and the distance $SB = AC$, therefore velocity in the ellipse at the mean distance : velocity in the circle at the same distance

$$:: \frac{L^{\frac{1}{2}}}{BC} : \frac{(2AC)^{\frac{1}{2}}}{AC} :: L^{\frac{1}{2}} : \left(\frac{2BC^2}{AC}\right)^{\frac{1}{2}},$$

therefore the velocities are equal.

Cor. 5. In the same figure, or in different figures having their latera recta equal, the velocity varies inversely as the perpendicular from the focus on the tangent.

Cor. 6. In the parabola, the velocity varies in the inverse subduplicate ratio of the distance of the body from the focus, in the ellipse it varies in a greater, and in the hyperbola in a less inverse ratio.

For the $(\text{velocity})^2 \propto \dfrac{1}{SY^2}$,

which in the parabola $\propto \dfrac{1}{SP}$,

$$\text{in the ellipse} \propto \frac{HP}{SP} \propto \frac{2AC - SP}{SP},$$

$$\text{in the hyperbola} \propto \frac{HP}{SP} \propto \frac{2AC + SP}{SP}.$$

Cor. 7. In the parabola, the velocity of the body at any distance from the focus is to the velocity of a body revolving in a circle at the same distance from the center, in the subduplicate ratio of 2 : 1; in the ellipse it is less, in the hyperbola greater than in this ratio.

For, velocity in the conic section : velocity in the circle at the same distance

$$:: \frac{L^{\frac{1}{2}}}{SY} : \frac{(2SP)^{\frac{1}{2}}}{SP} :: \left(\frac{L \,.\, SP}{2SY^2}\right)^{\frac{1}{2}} : 1 :: \sqrt{2} : 1$$

$$\text{in the parabola} :: \left(\frac{BC^2 \,.\, SP}{AC \,.\, SY^2}\right)^{\frac{1}{2}} : 1 :: \left(\frac{HP}{AC}\right)^{\frac{1}{2}} : 1$$

in the ellipse or hyperbola, and $HP < 2AC$ in the ellipse, and $> 2AC$ in the hyperbola.

Hence also, in the parabola, the velocity is everywhere equal to the velocity in a circle at half the distance, in the ellipse less, and in the hyperbola greater.

COR. 8. The velocity of a body revolving in any conic section, is to the velocity in a circle at the distance of half the latus rectum, as that distance is to the perpendicular from the focus on the tangent.

For, velocity in conic section : velocity in the circle at distance $\frac{1}{2}L :: \frac{L^{\frac{3}{2}}}{SY} : \frac{L^{\frac{3}{2}}}{\frac{1}{2}L} :: \frac{1}{2}L : SY.$

COR. 9. Hence, since (Cor. 6, Prop. IV.) the velocity of a body revolving in a circle is to the velocity in any other circle in the inverse subduplicate ratio of the distances, the velocity of a body in a conic section will be to the velocity in a circle at the same distance, as a mean proportional between that common distance and the semilatus rectum to the perpendicular from the focus on the tangent.

For velocity in a circle at distance $\frac{1}{2}L$ : velocity in a circle at distance $SP :: SP^{\frac{1}{2}} : (\frac{1}{2}L)^{\frac{1}{2}}$, therefore velocity in conic section : velocity in circle at distance $SP$

$$:: (\tfrac{1}{2}L \,.\, SP)^{\frac{1}{2}} : SY.$$

---

167. *To find the velocity in a conic section described under the action of a force tending to the focus.*

In the central conic sections

$$V^2 = \frac{h^2}{SY^2} = \frac{\mu BC^2}{AC \,.\, SY^2} = \frac{\mu HP}{AC.SP},$$

$$\text{or, } V^2 = F.\tfrac{1}{2}PV = \frac{\mu}{SP^2}.\frac{CD^2}{AC} = \frac{\mu \,.\, HP}{SP \,.\, AC}.$$

In the parabola,

$$V^2 = \frac{h^2}{SY^2} = \frac{\mu \,.\, 2SA}{SA \,.\, SP} = \frac{2\mu}{SP},$$

$$\text{or, } V^2 = F \cdot \tfrac{1}{2} PV = \frac{\mu}{SP^2} \cdot 2SP = \frac{2\mu}{SP},$$

$$HP = 2AC - SP, \text{ in the ellipse,}$$

$$HP = SP - 2AC, \text{ in the hyperbola, force repulsive,}$$

$$= SP + 2AC, \text{ in the hyperbola, force attractive;}$$

$$\therefore\ V^2 = \frac{\mu}{SP}\left(2 \mp \frac{SP}{AC}\right).$$

168. The expression $\frac{\mu}{SP}\left(2 - \frac{SP}{AC}\right)$ for the square of the velocity in the ellipse, reduces itself to that for the hyperbola under an attractive force by changing the sign of $AC$, which corresponds to the opposite direction in which $AC$ is measured in the hyperbola; it reduces to that for the hyperbola under a repulsive force by changing the sign of $\mu$, which corresponds to changing the direction of the force; and to that for the parabola by making $AC$ infinite.

169. *To compare the velocity in the ellipse or hyperbola with that in the circle at the same distance.*

Let $U$ be the velocity in the circle,

$$U^2 = \frac{\mu}{SP^2} \cdot SP = \frac{\mu}{SP};$$

$$\therefore\ V^2 : U^2 :: 2 \mp \frac{SP}{AC} : 1,$$

$$V = U\sqrt{2 \mp \frac{SP}{AC}}.$$

*The Hodograph.*

170. Def. If from any point lines be drawn representing in direction and magnitude the velocity of a particle describing an orbit under the action of a force tending to a fixed center, the locus of the extremities of these lines is the *Hodograph*.

This name is given to the curve by Sir William Hamilton, see page 613 of his work on Quaternions.

171. PROP. *If a conic section be described under the action of a force tending to a focus, the Hodograph is a circle.*

For in the case of the ellipse or hyperbola the velocity varies inversely as $SY$, and therefore directly as $HZ$, and the locus of $Z$ is a circle. And, in the case of a parabola, $AY$ being the tangent at the vertex, $AU$ perpendicular to $SY$,

$$SY : AS :: AS : SU,$$

therefore $SU$ varies as the velocity, and the locus of $U$ is a circle. Whence the proposition follows.

*Illustrations.*

1. The hodograph for an ellipse about the center is a similar ellipse.

For $CD$ is parallel to the direction of motion and proportional to the velocity.

2. The hodograph for a hyperbola is a hyperbola similar to the conjugate hyperbola.

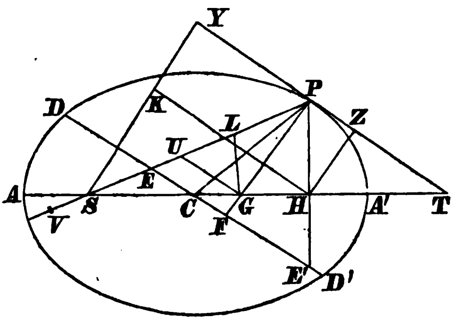
3. The hodograph for a parabola described under the action of a constant force parallel to the axis is a straight line parallel to the axis.

## PROP. XVII. PROBLEM IX.

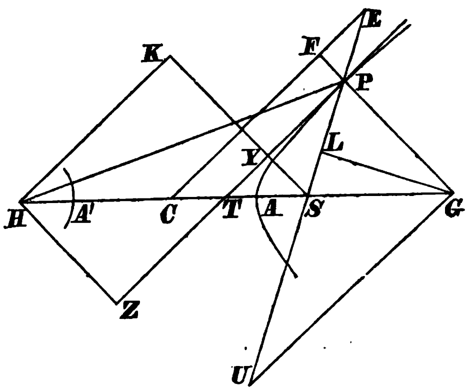
*Given that the centripetal force is inversely proportional to the square of the distance from the center, and that the absolute force of the center is known; it is required to find the curve which will be described by a body which is projected from a given point with a given velocity in a given direction.*

*Lemma in Conic Sections.*

In any conic section, if $G$ be the intersection of the axis and normal at $P$, $GL$ the perpendicular on the focal distance $SP$, $GU$ parallel to the tangent meeting $SP$ in $U$, $PL$ is the semi-latus rectum, and $PU$ half the chord of curvature.




1. In the ellipse and hyperbola, let $PG$ meet the conjugate diameter in $F$;




$\therefore\ CD.PF = AC.BC,$

$PG.PF = BC^2;$

$$\therefore\ \frac{PU}{PG} = \frac{PE}{PF} = \frac{CD}{BC};$$

$$\therefore\ \frac{PU}{CD} = \frac{PG}{BC} = \frac{BC}{PF} = \frac{CD}{AC};$$

$$\therefore\ PU = \frac{CD^2}{AC} = \text{half the chord of curvature,}$$

and $\dfrac{PL}{PG} = \dfrac{PF}{PE}$; $\therefore\ PL = \dfrac{BC^2}{AC}$ = half the latus rectum.

2. In the parabola,

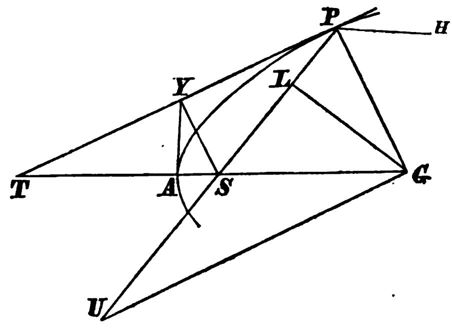
$$\frac{PU}{PG}=\frac{SP}{SY},$$

and $PG=2SY$;

$\therefore\ PU=2SP$.

Also, $\dfrac{PL}{PG}=\dfrac{SY}{SP}$;

$\therefore\ PL=\dfrac{2SY^2}{SP}=2SA$.

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Let $V$ be the velocity, $PY$ the direction of projection, $S$ the point to which the force tends, and let $PU$ be measured on $PS$ produced, if necessary, equal to twice the space through which the body must be drawn from rest by the action of the force at $P$ continued constant, in order that the velocity $V$ may be generated; therefore since the absolute force is given, $PU$ is given. Draw $PG$ perpendicular to $PY$ and $PH$ so that the angles $HPG$, $SPG$ are equal. Draw $UG$ perpendicular to $PG$, and join $SG$.

Here three cases arise:

1. If $PU=2SP$, $S$ is the center of a circle about $PGU$, $\therefore\ \angle SGP=\angle SPG=\angle HPG$, $\therefore\ SG$ produced either way will not meet $PH$.

2. If $PU<2SP$, $\angle SGP>\angle SPG$ or $HPG$, $\therefore\ SG$ produced meets $PH$ in $H$.

3. If $PU>2SP$, $\angle SGP<\angle SPG$ or $HPG$, $\therefore\ GS$ produced meets $PH$ in $H$.

1. If $PU = 2SP$ draw $GL$ perpendicular to $PS$, and with $S$ as focus, and $2PL$ as latus rectum describe a parabola whose axis is in the direction $SG$.

Then $PU$, by the Lemma, is the half chord of curvature at $P$ through $S$.

2. If $PU < 2SP$ with $S$ and $H$ as foci and $SP + HP$ as major-axis, describe an ellipse, then, by the Lemma, $PU$ is the half chord of curvature at $P$ through $S$.

3. If $PU > 2SP$ with $S$ and $H$ as foci $HP - SP$ as transverse axis, describe a hyperbola, $PU$ is the half chord of curvature through $S$.

In all these cases, a body may be supposed to revolve in the conic section described, under the action of the force tending to $S$ (Art. 139, 140), and the velocity at $P$ is that due to falling through one-fourth of the chord of curvature through $S$ or half $PU$, under the action of the force at $P$ supposed constant, and is therefore equal to $V$, the velocity of the projected body; also, since the angles $SPG$, $HPG$ are equal, $PY$ is a tangent, therefore the direction of motion is that of the projected body.

Therefore the circumstances of the two bodies are the same in all respects which can influence the motion at the point $P$, and they will therefore describe the same orbits, or the projected body will describe a conic section of that kind which corresponds to the velocity.

The orbit therefore is an ellipse, parabola, or hyperbola, according as $PU <, =, \text{ or } > 2SP$,

$$\text{or as } V^2 (= F.PU) <, =, \text{ or } > 2F.SP,$$

$$\text{or as } V^2 <, =, \text{ or } > 2 \times (\text{vel.})^2 \text{ in a circle of radius } SP.$$

Cor. 1. Hence if a body move in any conic section, and be disturbed by any impulse from its orbit, the orbit in

which it will proceed to move may be discovered. For, by compounding the motion of the body with that motion which the impulse alone would generate, the motion and direction of motion will be found with which the body will proceed from the point at which the disturbance took place.

COR. 2. And if the body be disturbed by any continuous extraneous force, its course can be determined, approximately, by calculating the changes which the force produces at certain points, and estimating from analogy the changes which take place at the intermediate points.

### SCHOLIUM.

If a body $P$ move in the perimeter of any conic section, whose center is $C$, under the action of a centripetal force tending to any given point $R$, and the law of force be required, draw $CG$ parallel to $RP$ and meeting in $G$ the tangent $PG$ to the conic section.

Then, by Prop. VII. Cor. 3, force tending to $R$ : force tending to $C$ :: $CG^3 : CP \,.\, RP^2$, but, force tending to $C$ varies as $CP$,

$$\therefore \text{ force tending to } R \propto \frac{CG^3}{RP^2}.$$

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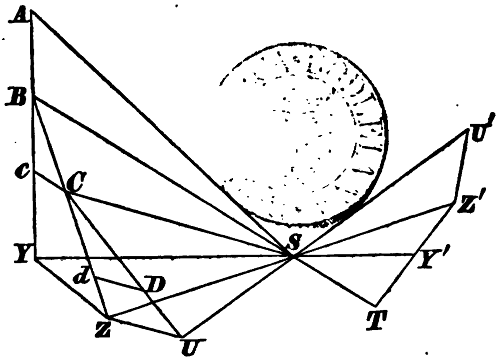
172. An elegant direct investigation of the path of a body projected at any angle to the radius drawn to a given center, to which a force tends which varies inversely as the square of the distance, is given in *Goodwin's Course of Mathematics*, being due to R. L. Ellis, Esq. of Trinity College; in this investigation the properties of the hodograph are introduced, and the path is shewn to be the locus of a point whose distance from a fixed

straight line is in a constant ratio to its distance from the center of force.

For the outlines of the following demonstration, also depending on the properties of the hodograph, I am indebted to Professor Tait, of Belfast, to whom I proposed the problem to shew that the feet of the perpendiculars from the center of force on the direction of motion of the projected body always lie in a circle or straight line.

173. *General property of the motion of a body in a central orbit.*

Let $ABC$ be a portion of a polygonal perimeter described under the action of impulses tending to $S$ as in Prop. I.




$SY$, $SZ$, perpendicular to $AB$, $BC$; produce $YS$, $ZS$ to $Y'Z'$ making $YS \,.\, SY' = ZS \,.\, SZ'$.

Then $SY'$, $SZ'$ represent the velocities in $AB$, $BC$ in magnitude, and are perpendicular to the directions of motion;

$$\therefore\ SY' : SZ' :: Bc : BC,$$

$$\text{and } \angle\, Y'SZ' = \angle\, YSZ = \angle\, YBZ;$$

$$\therefore \text{ triangles } cBC,\ Y'SZ' \text{ are similar,}$$

and $Y'Z'$ is perpendicular to $BS$ produced.

Also if $Y'Z'U'\ldots$ be the polygon corresponding to $ABCD\ldots$ making the same construction for each side successively,

$$Y'Z' : Cc :: Z'U' : Dd :: \ldots\ldots$$

$\therefore$ the perimeter $Y'Z'U'\ldots$ varies as the sum of the velocities generated by the impulses in the corresponding portion of the perimeter of the original polygon.

If we proceed to the limit we obtain the following results:

1. If a body describe any curve under the action of a force tending to $S$, and $YS$ perpendicular to the tangent on any point $P$ be produced to $Y'$, so that $SY'.SY$ is invariable, the tangent to the locus of $Y'$ is perpendicular to $PS$.

2. Any finite arc of the locus of $Y'$ varies as the sum of the velocities generated by the central force in the passage through the corresponding arc of the trajectory.

3. The chord of the arc represents the resultant of the velocities generated by the central force, and is perpendicular to its direction.

174. *To shew that if the central force vary inversely as the square of the distance, a body will describe a conic section.*

The velocity generated in a small given time varies ultimately inversely as the square of the distance, also the angle described in the same time varies ultimately inversely as the square of the distance, $\therefore$ the velocity generated varies as the angle described; hence, by Lemma IV., the velocity generated in a finite time varies as the whole angle described.

Now by result (1) of the last proposition, the angle described is equal to the angle between the tangents at the extremities of the corresponding arc of the locus of $Y'$, and by (2) this velocity varies as the arc of that locus; $\therefore$ the locus is such that the angle between the tangents at the extremities

of *any* arc varies as the arc, which is a property peculiar to the circle.

Hence the foot of the perpendiculars from the center of force on a tangent to the body's path is a circle or straight line, which is a property of a conic section only, since straight lines drawn according to a fixed law can only have one envelope.

It is easily seen that the path will be an ellipse, parabola, or hyperbola according as $S$ lies within, upon, or without the perimeter of the locus of $Y'$.

175. *Equations for determining the elements of the elliptic orbit, when* $V^2 < \frac{2\mu}{SP}$.

Let $V$ be the velocity of projection, $\alpha$ the angle $SPY$ between $SP$ and $PY$, fig. page 216, the direction of projection, $\mu$ the absolute force, $\psi$ the angle $PTS$ between $PY$ and the major-axis, and let $SP = R$, $a$, $b$, $e$ the semiaxes and eccentricity of the orbit;

$$\therefore\ V^2 = \frac{\mu HP}{SP \cdot AC} = \frac{\mu}{R}\left(2 - \frac{R}{a}\right) \quad . \quad . \quad . \quad (1)$$

$$\mu \cdot \tfrac{1}{2}L = h^2 = V^2R^2\sin^2\alpha;$$

$$\therefore\ \frac{b^2}{a} = a(1-e^2) = \frac{V^2R^2\sin^2\alpha}{\mu} \quad . \quad . \quad . \quad . \quad (2)$$

Draw $SY$, $HZ$ perpendicular upon the tangent, and $HK$ perpendicular to $SY$,

$$SH\cos SHK = HK = YZ = (SP + PH)\cos SPY;$$

$$\therefore\ 2ae\cos\psi = 2a\cos\alpha;$$

$$\therefore\ e\cos\psi = \cos\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$SH\sin SHK = SK = SY - HZ;$$

$$\therefore\ 2ae\sin\psi = (SP - HP)\sin\alpha$$

$$= \{R - (2a - R)\}\sin\alpha;$$

$$\therefore\ e \sin \psi = \left(\frac{R}{a} - 1\right) \sin \alpha\,;$$

$$\therefore\ \tan \psi = \left(\frac{R}{a} - 1\right) \tan \alpha$$

$$= \left(1 - \frac{RV^2}{\mu}\right) \tan \alpha. \quad . \quad . \quad . \quad (4)$$

The equations (1) and (2) determine $a$ and $b$ or $e$, and (4) determine $\psi$ immediately from the given circumstances of projection, (3) is also a convenient equation for determining the position of the axes.

Instead of (3) or (4) we might employ the equation

$$R = \frac{\frac{1}{2}L}{1 + e \cos ASP}$$

to determine the angle $ASP$, which gives the direction of the axes.

176. *Equations for determining the elements of the hyperbolic orbit, when* $V^2 > \dfrac{2\mu}{SP}$.

$$V^2 = \frac{\mu}{R}\left(2 + \frac{R}{a}\right) \quad . \quad . \quad . \quad . \quad (1)$$

$$\mu\,\frac{b^2}{a} = \mu a\,(e^2 - 1) = V^2R^2 \sin^2 \alpha\,. \quad . \quad (2)$$

and $SH \cos SHK = HK = YZ = (HP - SP) \cos \alpha\,;$

$$\therefore\ e \cos \psi = \cos \alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Also, $SH \sin SHK = SK = SY + YK\,;$

$$\therefore\ e \sin \psi = \{R + (2a + R)\} \sin \alpha\,;$$

$$\therefore\ \tan \psi = \left(\frac{R}{a} + 1\right) \tan \alpha$$

$$= \left(\frac{RV^2}{\mu} - 1\right) \tan \alpha. \quad . \quad . \quad . \quad (4)$$

177. *Equations for determining the elements of the parabolic orbit, when* $V^2 = \dfrac{2\mu}{SP}$.

In fig. page 217, $SY^2 = AS \,.\, SP$;

$$\therefore\ AS = R\sin^2 a,\ (1)\ \text{ and }\ PTS = 2a,\ (2)$$

(1) and (2) are equations which completely determine the position and dimensions of the orbit.

178. *To find the elements of the orbit described under the action of a repulsive force varying inversely as the square of the distance from the point from which the force tends.*

Let $H$ be the point from which the force tends, $HP = R$,

$$V^2 = \frac{\mu SP}{HP\,.\,AC} = \frac{\mu}{HP}\,\frac{HP - 2AC}{AC} = \frac{\mu}{R}\left(\frac{R}{a} - 2\right) \quad . \quad (1)$$

The other equations are similar to those in Art. 176.

*Illustrations.*

1. A body is revolving in a circle under the action of a force which tends to the center and varies inversely as the square of the distance from it. When the body arrives at any point, if the force begin to tend to the point of bisection of the radius through the body, to determine the orbit described by the body.

Let $CA$ be the radius, $S$ the new center of force. Then since the force is finite, the velocity at $A$ is unaltered, and $A$ is an apse of the new orbit.

Also (velocity)$^2$ in the circle $= \dfrac{\mu}{CA^2}\,.\,CA = \dfrac{\mu}{CA} < \dfrac{2\mu}{SA}$; hence, the body moves in an ellipse, and $\dfrac{\mu}{CA} = \dfrac{\mu}{SA}\left(2 - \dfrac{SA}{a}\right)$.................(1)

$$\therefore\ a = \frac{2}{3} SA = \frac{1}{3} CA,$$

$$\text{and } \mu \frac{b^2}{a} = h^2 = \frac{\mu}{CA} . SA^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$\therefore\ b^2 = \frac{1}{4} . CA . a = \frac{1}{12} CA^2.$$

$$\text{Also } 1 - \frac{b^2}{a^2} = 1 - \frac{3}{4} = \frac{1}{4}; \quad \therefore\ e = \frac{1}{2}.$$

*Aliter.*

Instead of equation (2) we might determine $e$ from the consideration that $A$ was one extremity of the major-axis;

$$\therefore\ SA = a(1 \pm e);$$

$$\therefore\ 1 \pm e = \frac{3}{2};$$

$\therefore\ e = \frac{1}{2}$, and the upper sign must be taken, or $A$ is the greatest focal distance.

The orbit lies entirely within the circle, since the force at $A$ is increased, and therefore the curvature greater than that in the circle.

2. **If the new center of force be in the bisection of the radius which *if produced* passes through the body; to determine the orbit.**

$$\frac{\mu}{CA} = \frac{\mu}{SA}\left(2 - \frac{SA}{a}\right),$$

for the orbit must be elliptic since $\frac{SA}{CA} = \frac{3}{2} < 2$;

$$\therefore\ \frac{SA}{a} = 2 - \frac{3}{2} = \frac{1}{2}; \quad \therefore\ a = 3\,CA;$$

$$\text{and } SA = a(1 \pm e); \quad \therefore\ e = \frac{1}{2},$$

and $A$ is the nearest point to $S$.

In this case the force, and therefore the curvature, is diminished, which accounts for the orbit being exterior to the circle.

3. A particle is projected round a center of force which varies inversely as the square of the distance, with a velocity which is to the velocity in a circle at the same distance as $\sqrt{5} : 2$, and at an angle whose sine is $\frac{2}{\sqrt{5}}$; shew that the eccentricity of the orbit is $\frac{1}{2}$ and that the major-axis is perpendicular to the distance of projection.

$$V^2 = \frac{5}{4} \cdot \frac{\mu}{R} = \frac{\mu}{R}\left(2 - \frac{R}{a}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\mu a (1 - e^2) = V^2 . R^2 . \frac{4}{5} = \mu R \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$\therefore\ a = \frac{4}{3} R,$$

$$1 - e^2 = \frac{3}{4}; \quad \therefore\ e = \frac{1}{2},$$

$$\text{and } e \cos \psi = \cos \alpha \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$\therefore\ \frac{1}{2} \cos \psi = \sqrt{1 - \frac{4}{5}} = \frac{1}{\sqrt{5}};$$

$$\therefore\ \cos \psi = \frac{2}{\sqrt{5}} = \sin \alpha;$$

$\therefore\ \psi$, the angle between the direction of projection and major-axis, $= \frac{\pi}{2} - \alpha$, or the major-axis is perpendicular to the distance.

4. A body revolves in a circle under the action of a force tending to the center and varying inversely as the square of the distance. Find the orbit described if the force suddenly tend to a point $S$ in the circumference of the circle $60^0$ from the body.

(velocity)$^2$ at $A = \frac{\mu}{CA} = \frac{\mu}{SA}$, and since the velocity is unaltered at $A$,

$$\frac{\mu}{SA} = \frac{\mu}{SA}\left(2 - \frac{SA}{a}\right);$$

$\therefore\ a = SA$, or $A$ is the extremity of the minor-axis; $\therefore$ the major-axis

is parallel to the tangent at $A$, or perpendicular to $CA$, and the center is in the bisection of $CA$.

The curvature is less than that of the circle, because the normal force is diminished by the change.

5. A body revolving in an ellipse under the action of a force tending to a focus $S$ has the direction of its motion altered at a given point of its path, the velocity remaining unaltered; to determine the corresponding change in the position of the major-axis.

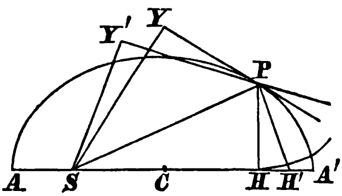
Since the velocity and the distance $SP$ in the new orbit is the same as in the old, $\therefore$ the length of the major-axis is the same; $\therefore$ $PH$ is the same in the two orbits; $\therefore$ the other focus lies in a circle whose center is $P$, and $SP$, $PH$ make equal angles with the new direction.

6. To find at what point of the orbit a slight alteration may be made in the direction of motion, the velocity remaining unaltered, so that the direction of the major-axis may be the same as before.

The direction of the major-axis being unaltered, $SH$ must be a tangent to the locus of $H$, $\therefore$ $P$ must be at the extremity of the latus rectum which does not contain the center of force.

7. Prove that if, when a body is at the extremity of the latus rectum which does not contain the center of force, the direction of motion is deflected through a small angle, without altering the velocity, the alteration of the eccentricity is to the deflection as $BC^2 : AC^2$.

For, let $P$ be the position of the body, $HH'$ the small arc of the circle described by $H$, which nearly coincides with the direction of the major-axis, $HPH'$ is double the angle of deflection, and $\frac{H'S}{2AC} - \frac{HS}{2AC}$, or $\frac{HH'}{2AC}$, is the change of eccentricity;



$\therefore$ change of eccentricity : deflection of direction

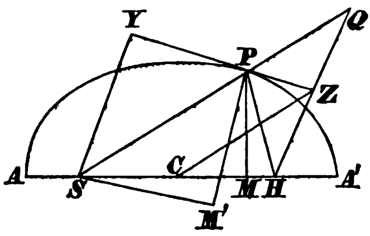
$$:: \frac{HH'}{2AC} : \frac{HH'}{2HP} :: HP : AC :: BC^2 : AC^2.$$

8. If a body moving in an ellipse be acted on by an impulsive force in the direction of the focus, when it arrives at the extremity of the latus rectum, the axis-major remains unaltered in direction.

For if $SL$ be the semi latus rectum, the force being central, $h$ is unaltered; $\therefore \mu . SL$ is unaltered, or $SL$ is the semi latus rectum of the new orbit, and the axis-major is perpendicular to $SL$.

9. The velocity at any point of an ellipse about a force in the focus is compounded of two uniform velocities, one $= \frac{\mu}{h}$ perpendicular to the radius vector, and the other $= \frac{\mu e}{h}$ perpendicular to the major-axis.

Let $S$ be the center of force, $HZ$ perpendicular on the tangent at $P$, join $CZ$. Then $ZC$, parallel to $PS$, $CH$ and $HZ$ are perpendicular to the three directions; $\therefore$ velocity represented by $HZ$ is the resultant of two represented by $CZ$ and $HC$ in magnitude; now the velocity perpendicular to $HZ = \frac{h}{SY}$ $= \frac{h}{b^2} . HZ$; $\therefore$ velocity perpendicular to $HC$ and $CZ$ are $\frac{h}{b^2} ae$, and $\frac{h}{b^2} a$ $= \frac{\mu e}{h}$ and $\frac{\mu}{h}$, since $\mu \frac{b^2}{a} = h^2$.




10. A particle moving in an ellipse under the action of a force tending to the focus has a very small velocity $\frac{n\mu}{h}$ impressed upon it in the direction of the radius vector; shew

that the corresponding changes of the eccentricity and angular distance of the apse are given by the equations

$$e' - e = n \sin \theta,$$
$$e(\theta' - \theta) = n \cos \theta.$$

For, since the impressed velocity is towards $S$, $\therefore \frac{\mu}{h}$ in the new orbit is still the velocity perpendicular to the radius vector: and the remaining velocity $\frac{e'\mu}{h}$ is compounded of the two velocities $\frac{e\mu}{h}$ in direction $PM$, and $\frac{n\mu}{h}$ in $PS$, let $PM'$ be the perpendicular on the new major-axis. Then $\angle M'SM = \angle M'PM$ being angles in the same arc of a circle about $SPM$, and the velocity in $PM'$ and its components in $PM$ and $PS$ being as $e'$, $e$ and $n$,

$$e' \sin M'PM = n \sin SPM,$$
$$e' \cos M'PM = e + n \cos SPM;$$

$\therefore$ since $M'PM = \theta' - \theta$ is small, and $SPM = 90^0 - \theta$, this proposition is proved.

## XVIII.

1. The velocity in an ellipse at the greatest distance is half that with which a body would move in a parabola at the same distance; required the eccentricity of the ellipse.

2. A body, moving in a parabola about a center of force in the focus, meets at the vertex with an obstacle which diminishes the square of the velocity by one fourth, without altering the direction of the motion; shew that the body will afterwards move in an ellipse whose axis-major is equal to the latus rectum of the parabola.

3. If from every point of a hyperbola described under the action of a force in the farther focus a particle moved from rest under the action of the force at that point continued constant, until it acquired the velocity of the particle moving in the

hyperbola; find the locus of the particles. If $r$, $r'$ be the radii vectores for the hyperbola and locus, $2ar' = r^2$.

4. A body revolves in an ellipse about a center of force in the focus $S$. Shew that there is always some determinate point at which the absolute force may be supposed to change suddenly from $\mu$ to $n\mu$, so that the subsequent path of the body may be a parabola about $S$ in the focus, provided $n$ is not situated beyond the limits $\frac{1}{2}(1+e)$ and $\frac{1}{2}(1-e)$. Prove also that the latus rectum of the ellipse : that of the parabola :: $n$ : 1.

5. Of all comets moving in the ecliptic in parabolic orbits, that which has the latus rectum of its orbit equal to the diameter of the Earth's orbit will remain within the latter for the longest period, the Earth's orbit being considered circular.

6. The ratio of the axes of the Earth's and Venus's orbits is 18 : 13; find the periodic time of Venus.

7. Force $\propto$ (dist.)$^{-2}$; a body is projected with a velocity of 100 feet per minute from an initial distance of 32 feet, the velocity in a circle at that distance being 80 feet per minute; find the periodic time.

8. Two ellipses are described by two particles about the same center of force in the focus: the eccentricities are $\frac{1}{2}$ and $\frac{1}{2}\sqrt{3}$ respectively, and the major-axes are coincident in direction and equal in length. Compare the times which each body spends within the orbit of the other.

9. A body is moving in a given parabola under the action of a force in the focus: when it comes to a distance equal to the latus rectum, the force is suddenly changed into a repulsive one; determine the nature, position, and dimensions of the new orbit.

10. If a body be projected with a given velocity about a center of force which $\propto \dfrac{1}{(\text{dist.})^2}$, shew that the axis-minor of the

orbit described will vary as the perpendicular from the center of force upon the direction of projection; and determine the locus of the center of the orbit described.

11. A body moves in an ellipse and is at the extremity of the minor-axis when its velocity is doubled. Find the new orbit, and shew that the body will come to the vertex after describing $90^0$, provided the ratio of the axes of the given ellipse is $2 : 1$.

12. The velocity in a parabola round the focus is suddenly diminished in the ratio of $\sqrt{2} : 1$, shew that the semimajor-axis is $SP$, and the semiminor-axis is a mean proportional between $SP$ and $AS$.

13. A particle describes an ellipse about a center of force in the focus $S$; about $S$ as center a circle is described, which is cut by the radius vector $SP$ in the point $Q$; from $Q$ a line is drawn perpendicular to the direction of the particle's motion, which meets the major-axis in $R$; prove that $R$ is constant in position.

14. The perihelion distance of a comet moving in a parabolic orbit $= \frac{1}{2}$ the radius of the Earth's orbit, supposed circular. The planes of the orbits coinciding, find the time in days from perihelion to the point of intersection of the orbits.

15. If $PO$ is perpendicular on the directrix from any point of an elliptic orbit described by a particle about the focus $S$, and when the particle is at $P$, the force suddenly tends to $O$ instead of $S$, prove that the new orbit may be a parabola if $e > \frac{1}{3}$, and that in this case $SP$ passes through the intersection of the two circles, one described on $SH$ as diameter, and the other with center $S$ and radius $SA$, the shortest focal distance.

16. A body revolves in an ellipse about a center of force in its center $C$. When the body comes to $A$ the extremity of the axis-major, the law of the force is supposed to change suddenly

and $\propto D^{-2}$; find the elements of the new orbit. Also if the eccentricity of the old orbit be $e$, and that of the new orbit $e'$, then $e' = e^2$.

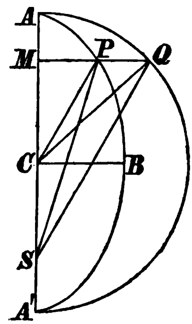
17. A body revolves in an ellipse about the focus from nearer to farther apse and the angle which its direction makes with the focal distance is constantly being increased without altering the velocity; shew that the motion of the apse line will change from progression to regression when the true anomaly $= \frac{\pi}{2} + 2\tan^{-1} e$, $e$ being the eccentricity at that moment.

18. A body describing an ellipse about a center of force in $S$ has a velocity equal to its own communicated in the direction $PH$, which causes it to describe a circle, determine the eccentricity of the original orbit, and shew that the diameter of the circle $= 4 \times$ the latus rectum.

# APPENDIX I.

1. To find the time of motion and the velocity acquired, when a body falls through a given space from rest, under the action of a force which varies inversely as the square of the distance from a fixed point.

Let $S$ be the center of force, $A$ the point from which the body begins to fall.




$\dfrac{\mu}{SP^2}$ the measure of the accelerating effect of the force at a distance $SP$.

Let $APA'$ be a semiellipse, whose focus is $S$ and axis-major $ASA'$, $AQA'$ the auxiliary circle whose center is $C$, $MPQ$ a common ordinate. Let a body revolving in the ellipse under the action of the force tending to $S$ arrive at $P$,

$$\text{time in } AP \propto \text{area } ASP \propto \text{area } ASQ;$$

$$\therefore \text{time in } AP : \text{time in } APA' :: \text{area } ASQ : \text{semicircle } AQA'$$
$$:: \text{sector } ACQ + \Delta SCQ : \text{semicircle } AQA';$$

$$\therefore \text{time in } AP = \frac{\pi AC^{\frac{3}{2}}}{\mu^{\frac{1}{2}}} \cdot \frac{AC \cdot \text{arc } AQ + SC \cdot QM}{\pi AC^2}.$$

This is true, whatever be the magnitude of the minor-axis $BC$, and therefore when it is indefinitely diminished, in which case the diameter of curvature at $A = \dfrac{2BC^2}{AC} = 0$, and therefore the velocity at $A = 0$; therefore the elliptic motion ultimately degenerates to a rectilinear motion in which the body starts from rest at $A$.

$$\text{Also, since } AS \,.\, SA' = BC^2,$$

$$SA' \text{ ultimately} = 0; \quad \therefore SC = AC = \tfrac{1}{2} SA;$$

$$\therefore \text{ time in } AM = \left(\frac{SA}{2\mu}\right)^{\frac{1}{2}} . (\text{arc } AQ + QM).$$

The velocity in the ellipse at

$$P = \left(\frac{\mu \,.\, (2AC - SP)}{AC \,.\, SP}\right)^{\frac{1}{2}},$$

and when the minor-axis is indefinitely diminished

$$\text{velocity at } M = \left(\frac{2\mu(AS - SM)}{AS \,.\, SM}\right)^{\frac{1}{2}} = \left(\frac{2\mu \,.\, AM}{AS \,.\, SM}\right)^{\frac{1}{2}}.$$

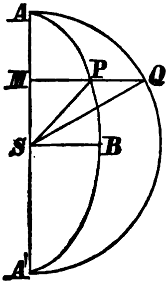
It must not be supposed that the motion will be represented throughout by the ultimate motion in an ellipse whose axis-minor is indefinitely diminished, in which case the body would return to $A$; for, since in this case the ellipse passes through $S$, we are precluded from applying the results of the second and third sections in determining the motion of the body after arriving at $S$; but we may correctly apply these results to determine the motion before arriving at $S$.

In order to determine the motion after arriving at $S$, we must observe that at $S$ there is a force in no direction, although when the body is at any point very near to $S$ there will be a very great force tending towards $S$; on approaching $S$, therefore, the velocity will continually increase and the body will pass through $S$, with very great velocity; but the motion will be checked, according to the same law, as rapidly as it was generated, and the body will proceed to a distance $SA$ on the opposite side of $S$.

Cor. If a body be projected directly towards or from a center, to which a force tends which varies inversely as the square of the distance, the time and velocity acquired in a given space may be determined by means of an ellipse, parabola, or hyperbola, whose latus rectum is indefinitely diminished, so constructed that at the point of projection the velocity is properly represented.

2. To find the time of motion and the velocity acquired when a body falls through a given space from rest, under the action of a force which varies as the distance from a fixed point.

Let $S$ be the center of force, $A$ the place from which the body begins to move; on $ASA'$ describe a semi-ellipse $APA'$ whose semiaxis-major is $SA$, and a semicircle $AQA'$, let $MPQ$ be a common ordinate.




Suppose a body revolving in the ellipse to arrive at $P$, then, time in $AP \propto$ area $ASP \propto$ sector $ASQ \propto$ angle $ASQ$;

$\therefore$ time in $AP$ : time in $ABA'$ :: arc $AQ : \pi . AS$;

$$\therefore \text{ time in } AP = \frac{\pi}{\sqrt{\mu}} \cdot \frac{\text{arc } AQ}{\pi . AS} = \frac{1}{\sqrt{\mu}} \times \frac{\text{arc } AQ}{AS},$$

and the same is true when the minor-axis is indefinitely diminished, in which case velocity at $A = 0$, since the diameter of curvature $= 0$.

Therefore the elliptic motion is reduced to the rectilinear motion of a body originally at rest at $A$, and the time in $AM$

$$\text{is } \frac{1}{\sqrt{\mu}} \times \frac{\text{arc } AQ}{AS}.$$

The velocity in the ellipse at $P$

$$= \sqrt{\mu} . SD \text{ where } SD \text{ is conjugate to } SP$$

$$= \sqrt{\mu}(AS^2 + BS^2 - SP^2)^{\frac{1}{2}};$$

therefore the velocity at $M$ in the rectilinear motion

$$= \sqrt{\mu}(AS^2 - SM^2)^{\frac{1}{2}} = \sqrt{\mu} . MQ.$$

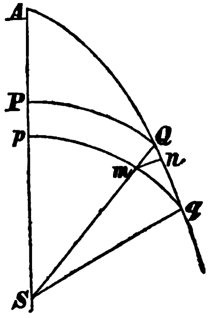
COR. Time from $A$ to $S = \dfrac{\pi}{2\sqrt{\mu}}$,

or the time of reaching $S$ is the same whatever be the initial distance.

For a direct solution, see Art. page 80, 81.

3. If the velocities of two bodies, one of which is falling directly towards a center of force and the other describing a curve about that center, be equal at any equal distances they will always be equal at equal distances, if the force depend only on the distance.

Let $S$ be the center of force, and let one of the bodies be moving in the straight line $APS$, the other in the curve $AQq$. Suppose the velocities at $P$, $Q$ to be equal, and let $Qq$ be an arc of the curve described in a short time. With center $S$ and radius $SQ$, $Sq$ describe circular arcs $QP$, $qP$ and let $SQ$ meet $pq$ in $m$ and draw $mn$ perpendicular to $Qq$.




Since the centripetal forces at equal distances are equal, they will be so at $P$ and $Q$, and $Pp$, $Qm$ may represent them; $Pp$ is wholly effective in accelerating $P$, $Qn$ is the only effective part of $Qq$ on $Q$, the component $nm$ being employed in retaining the body in the curve.

Also since the velocities are equal at $P$ and $Q$, the times of describing $Pp$, $Qq$ are ultimately proportional to $Pp$, $Qq$ when the time is indefinitely diminished;

$$\therefore \text{ force at } P : \text{force at } Q \text{ in } Qq :: Pp : Qn,$$

$$\text{time in } Pp : \text{time in } Qq :: Pp : Qq;$$

$$\therefore \text{vel}^y \text{ acquired at } p : \text{vel}^y \text{ acquired at } q :: Pp^2 : Qn . Qq;$$

$$\text{but } Qn . Qq = Qm^2 = Pp^2;$$

$\therefore$ velocity added in $Pp$ and $Qq$ are equal,

$\therefore$ the velocities at $p$ and $q$ are equal,

$\therefore$ if the velocities, &c.

# APPENDIX II.

## CYCLOID.

Def. If, in the same plane, a circle be conceived to roll along a straight line, any point on its circumference will describe a curve called a Cycloid.

Let $C$, $D$ be the points where the tracing point meets the straight line, $A$ the point where the tracing point $P$ is furthest from $CD$, $AB$ the corresponding diameter of the circle.

The revolving circle is called the *generating circle*, $AB$ is called the *axis*, $A$ the *vertex*, $CD$ the *base*.

1. If $RPS$ be the generating circle in any position, then, since the points of the base and circle come successively in contact $CS =$ arc $PS$, $CB$ and $BD$ are each half of the circumference of the circle, and $BS =$ arc $RP$.

2. *To draw a tangent to a cycloid.*

Let the generating circle be in the position $RPS$, then considering a circle as the limit of a regular polygon of a large number of sides, it will roll by turning about the point of contact, which is at rest for an instant, being the angular point of the polygon.

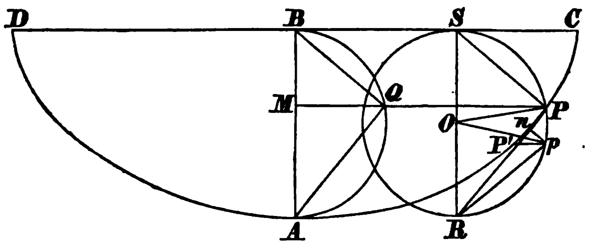
$\therefore$ $P$ moves perpendicular to $SP$, for an instant, or in the direction $PR$ of the supplemental chord, which is therefore the tangent at $P$.

Cor. If $AQB$ be the circle on $AB$ as diameter, $PQM$ an ordinate perpendicular to $AB$ the tangent at $P$ is parallel to the chord $QA$.

3. *To find the length of the arc of a cycloid.*

Let $RPS$ be the position of the generating circle corresponding to the point $P$ in the cycloid, let $P'$ be the position

of $P$, when the circle has turned through a small angle $POp$, so that $P'p$ is parallel to the base, and ultimately $PP'$ is perpendicular to $SP$, and $pP$ to $OP$;




∴ the triangles $PpP'$ and $OPS$ are similar;

∴ the triangle $PpP'$ is isosceles, and if $pn$ be drawn perpendicular to $RP$, $PP' = 2Pn = 2(RP - Rp)$ ultimately;

∴ the cycloidal arc from the vertex increases twice as fast as the supplemental chord, and they commence together,

$$\therefore \text{ arc } AP = 2RP = 2AQ.$$

4. *To find the relation between the arc and abscissa.*

Let $AM$ be the abscissa of the point $P$,

$$AM : AQ :: AQ : AB;$$

$$\therefore AP^2 = 4AQ^2 = 4AB.AM.$$

5. *To find the area of the cycloid.*

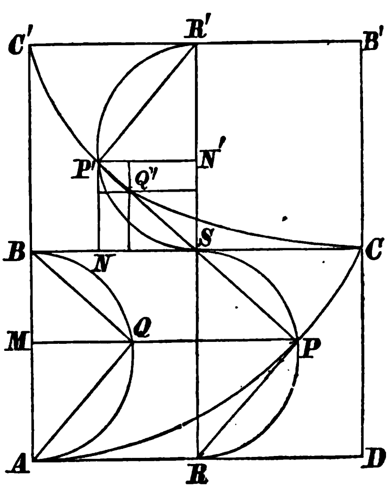
Let $P'$ (fig. Art. 7) be any point in the cycloid $CP'C'$, $P'S$ the chord of the generating circle which touches the cycloid, then $Q'$ a point in the cycloid near $P'$ ultimately coincides with $P'S$. Let $Q'N'$, $Q'N$ be the complements of the parallelogram whose diagonal is $P'S$, and sides parallel and perpendicular to the base, these are equal ultimately;

∴ by the fourth Lemma, the cycloidal area $CNP'$ = circular segment $SP'N'$.

6. Cor. The exterior portion $C'BC$ is equal to the area of the semicircle, and the whole parallelogram $BCB'C'$ is the rectangle under the diameter and circumference of the generating circle, and is equal to twice the area of the whole circle; therefore the cycloidal area $CC'B'$ is three times the area of the semicircle.

7. *To shew that the evolute of a given cycloid is an equal cycloid.*

Let $APC$ be half the given cycloid, $AB$ the axis, $A$ the vertex, and $BC$ the base. Produce $AB$ to $C'$, making $BC' = AB$, and complete the rectangle $BCB'C'$, and let the semicycloid $C'P'C$ be generated by a circle whose diameter is equal to that of the generating circle of the given cycloid; $C$ the vertex, and $B'C'$ the base of this cycloid.




Let $SPR$, $SP'R'$ be two positions of the respective generating circles, having their diameters $RS$, $SR'$ in the same straight line, $P$, $P'$ the corresponding points of the cycloids.

Join $SP$, $PR$ and $SP'$, $P'R'$.

By the mode of generation,

$$\text{arc } SP = SC, \text{ and arc } SPR = BC;$$

$$\therefore \text{ arc } PR = BS = C'R' = \text{arc } P'R';$$

$$\therefore \text{ angle } PSR = \text{angle } P'SR';$$

$$\therefore PSP' \text{ is a straight line.}$$

Also, arc $P'S$ = arc $PS$; $\therefore$ chd. $P'S$ = chd. $PS$;

$\therefore$ $P'SP = 2P'S = P'C$ the cycloidal arc;

and $P'SP$ touches the cycloid $C'PC$ at $P'$;

therefore, a string fixed to the cycloid at $C'$, and wrapped over the arc of the semicycloid, will when unwrapped have its extremity in the arc of the given cycloid; and if another equal semicycloid be described by the circle rolling on $B'C'$ produced, the extremity of the string wrapped on this curve will trace out the remainder of the given cycloid.

Thus a pendulum may be made to oscillate in a given cycloid.

8. *To find the time of oscillation of a heavy particle moving in a smooth cycloidal arc, whose axis is vertical.*

A direct method of solving this problem is given in page 85, but it can be solved by means of the proposition given in Appendix I. Art. 2.

For the particle being in any position $P$ is acted on by a force the measure of the accelerating effect of whose component in direction of the motion is

$$g \cdot \frac{RP}{RS} = \frac{g}{2RS} \cdot AP, \text{ and } \frac{g}{2RS}, \text{ or } \frac{g}{2AB} \text{ is constant.}$$

The acceleration at every point is the same as if the particle moved in a straight line under the action of a force varying as the distance tending to a point in the line.

$\therefore$ the time of falling from any point to $A$ is $\frac{\pi}{2}\sqrt{\frac{2AB}{g}}$,

and the time of an oscillation from rest to rest $= \pi\sqrt{\frac{2AB}{g}}$,

being the same for all arcs of vibration.

9. Cor. The length of the string which by the contrivance of the last Article makes a particle oscillate in this cycloid

is $2AB = l$ suppose; therefore the time of the oscillation of a pendulum of length $l = \pi\sqrt{\frac{l}{g}}$.

10. *To find the time of a very small oscillation of a simple pendulum suspended from a point.*

A simple pendulum is an imaginary pendulum consisting of a heavy particle called the *bob*, suspended from a point by means of a rod or string without weight.

In this case the pendulum describes the small arc of a circle which may be considered the same as a cycloidal arc the axis of which is half the distance of the bob from the point of suspension.

The time of oscillation from rest to rest is $\pi\sqrt{\frac{l}{g}}$.

11. *To count the number of oscillations made by a given pendulum in any long time.*

In consequence of the liability to error in counting a very great number of oscillations, since in the case of a seconds pendulum for each hour there would be 3600 oscillations, it becomes necessary to adopt some contrivance for diminishing the labour. For this purpose a pendulum is made which oscillates nearly in the same time as the given pendulum. These pendulums are then placed one before the other, so that two points near their lowest positions shall be in the field of view of a fixed microscope at the same time, and the time of exact coincidence in a certain position can be observed.

Suppose that after $n$ oscillations of the given pendulum they are again in exact coincidence close to the same position: if there be $m$ such coincidences in the whole time of observation, the number of oscillations in that time is $mn$, and the only labour has been to count the $n$ oscillations, and to estimate the number of the coincidences before the last one observed.

12. *To measure the accelerating effect of gravity by means of a pendulum.*

Let $g$ be the measure of this effect or the velocity generated by the force of gravity in a second.

Let $l$ be the length of a simple pendulum which makes $n$ oscillations in $m$ hours, then $\frac{3600m}{n}$ = number of seconds in one oscillation $= \pi\sqrt{\frac{l}{g}}$; $\therefore\ g = \frac{\pi^2 ln^2}{(3600)^2 m^2}$, in whatever unit of length $l$ is estimated.

This would be a very exact method of determining $g$, if we could form a simple pendulum; but it is impossible to do this, and it is only by calculations of a nature too difficult to be explained here that it can be shewn how to deduce the length of the simple pendulum, which would oscillate in the same time as a pendulum of a more complicated structure.

13. The seconds pendulum at any place is the simple pendulum which at the mean level of the sea at that place would oscillate in one second.

14. *To determine the height of a mountain by means of a seconds pendulum.*

Let $x$ be the height of the mountain above the mean level of the sea, $L$ the length of the seconds pendulum for that place, $a$ the Earth's radius, all expressed in feet; $n$ the number of oscillations lost by the pendulum in 24 hours.

$g$ the accelerating effect of gravity at the mean level of the sea.

$\therefore\ \frac{ga^2}{(a+x)^2}$ = that at the top of the mountain, supposing the earth composed of spherical layers; $\therefore$ the time of oscillation at the top is

$$\pi\sqrt{\frac{L}{g}\frac{(a+x)^2}{a^2}} = \frac{a+x}{a} \text{ in seconds,}$$

since $\pi\sqrt{\frac{L}{g}} = 1$;

$$\therefore\ (24\times 60\times 60-n)\frac{a+x}{a}=24\times 60\times 60,$$

$$1+\frac{x}{a}=\frac{24\times 60\times 60}{24\times 60\times 60-n},$$

$$\text{and } \frac{x}{a}=\frac{n}{24\times 60\times 60}+\frac{n^2}{(24\times 60\times 60)^2}\text{ nearly;}$$

$$\therefore\ \text{if } a=4000\times 1760\times 3,$$

$$x=\frac{4000\times 1760\times 3}{24\times 60\times 60}n+\ldots\ldots$$

$$=245n+\frac{245\,n^2}{24\times 60\times 60}\text{ nearly;}$$

$$=245n+{\cdot}0027\,.\,n^2.$$

If $n=10$,

the height $=2450{\cdot}27$ feet.

15. *To find the number of seconds lost in a day, in consequence of a slight error in the length of the seconds pendulum; and conversely.*

Let $N$ be the number of seconds in a day, $L$ the length of the seconds pendulum; $L+\lambda$ that of the incorrect pendulum; $N-n$ the number of oscillations in a day.

$$\therefore\ (N-n)\,\pi\sqrt{\frac{L+\lambda}{g}}=N.\,\pi\sqrt{\frac{L}{g}};$$

$$\therefore\ 1+\frac{\lambda}{L}=\frac{N^2}{(N-n)^2};$$

$$\therefore\ \frac{\lambda}{L}=\frac{2nN-n^2}{(N-n)^2},$$

$$\text{and } n=\frac{N\lambda}{2L}\text{ nearly;}$$

whence $n$ can be found from $\lambda$, or $\lambda$ from $n$.

## EQUIANGULAR SPIRAL.

DEF. 1. If a series of radii $SA$, $SB$, $SC$,... be drawn inclined at equal angles, and $AB$, $BC$, $CD$,... be drawn making equal angles $SAB$, $SBC$,... with these radii respectively, the curvilinear limit of the polygon $ABCD$..., when the equal angles $ASB$, $BSC$, ... are indefinitely diminished, is the *Equiangular Spiral.*

DEF. 2. If an indefinite line $SP$ revolve uniformly about a fixed point $S$, while another point $P$ advances or recedes on that line with a velocity which varies as the distance from $S$, it will trace out the *Equiangular* or *Logarithmic Spiral.*

The second definition follows immediately from the first, since, fig. page 30, $SA - SB : SB - SC :: SA : SB$, the triangles $SAB$, $SBC$, ...being similar.

Since the limiting positions of the sides of the polygon are those of tangents to the curve, the inclination of the tangents to the radii at any point is a constant angle; whence the equiangular spiral is the spiral which cuts all the radii drawn from a fixed point at a constant angle.

*To find the length of an arc of an equiangular spiral contained between two radii.*

Let $\alpha$ be the angle $SAB$,

and let $SB : SA :: \lambda : 1$ a constant ratio, $\lambda < 1$;

$$\therefore BC : AB :: CD : BC :: \ldots :: \lambda : 1;$$

$$\therefore AB + BC + \ldots : AB :: 1 + \lambda + \lambda^2 + \ldots : 1,$$

$$:: 1 - \lambda^n : 1 - \lambda$$

$$:: SA(1 - \lambda^n) : SA - SB;$$

$\therefore$ proceeding to the limit, since $SL = \lambda^n . SA$,

$$\text{arc } AL : SA - SL :: AB : SA - SB \text{ ultimately;}$$

$$\therefore \text{arc } AL = (SA - SL) \sec \alpha.$$

Obs. $1-\lambda^{2m} : 1-\lambda^2 :: SA^2-SL^2 : SA^2-SB^2$;

$\therefore$ ultimately, $1-\lambda^{2m} : 1-\lambda^2 :: SA^2-SL^2 : 2SA\,.\,AB\cos\alpha$;

$\therefore$ in page 30, $AB^2+BC^2+\ldots\ldots : SA^2-SL^2 :: AB : 2SA\cos\alpha$.

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## CATENARY.

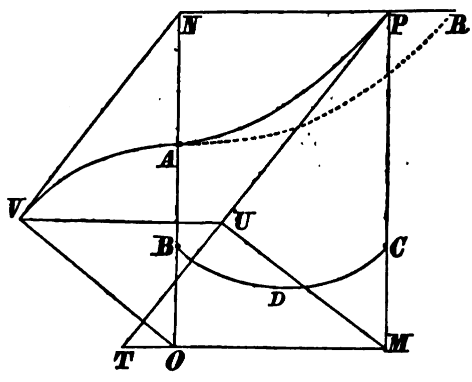
DEF. The *catenary* is the curve in which a uniform and perfectly flexible string, of which the extremities are suspended at two points, would hang under the action of gravity, supposed to be a constant force acting in parallel lines.

The *directrix* is a horizontal straight line whose depth below the lowest point is equal to the length of string whose weight is equal to the tension at the lowest point.

The *axis* is the vertical through the lowest point.

1. *The tension at any point of the catenary is equal to the weight of the string which if suspended from that point would extend to the directrix.*

Let $A$ be the lowest point of a uniform and perfectly flexible string hanging from two points under the action of gravity,




$P$ any other point, $AO$ the length of string whose weight is equal to the tension of the string at $A$.

Take a point $B$ in $OA$, and let $OM$, $BC$ drawn horizontally meet a vertical $PM$ in $M$ and $C$.

If a string pass round pegs at $APCB$, it is evident that it will rest in any position, and this will be the case whatever be the size of $BDC$, and therefore if $BDC$ be such that the tension at $B$ equal the weight of $BO$, therefore replacing the portion $BDC$ by strings $BO$ and $CM$, $OAPM$ is in equilibrium, but $AO$ has weight equal to the tension in the proposed case, therefore the shape of $AP$ in both cases is the same and the tension at $P$ is the weight at $PM$.

2. *If a circle be drawn on the ordinate perpendicular to the directrix as diameter, it will meet the tangent at a point whose distance from the point of contact is equal to the arc of the catenary.*

Let $PT$ be the tangent, $TOM$ the directrix, then since the arc $AP$ supposed to become rigid is kept at rest by the tensions at $A$ and $P$, parallel to $MT$, $TP$ and the weight parallel to $PM$, $TPM$ is a triangle of forces;

$$\therefore \text{ weight of } AP : \text{tension at } P :: PM : PT;$$

$$\therefore AP : PM :: PM : PT,$$

and if $MU$ be perpendicular to $PT$,

$$MU : PM :: PU : PM;$$

$$\therefore PU = AP.$$

Cor. Tension at $A$ : weight of $AP :: MT : PM$;

$$\therefore AO : PU :: MT : PM :: MU : PU;$$

$$\therefore AO = MU.$$

3. *To draw a tangent to a catenary at any point.*

With center $O$, and radius $OA$, describe a circle $AV$, draw $PN$ horizontal meeting the axis in $N$, and $NV$ touching the circle in $V$, $PT$ parallel to $NV$ is a tangent to the catenary at $P$.

For, join $OV$, and draw $MU$ perpendicular to $PT$, therefore $OV$ is equal and parallel to $MU$;

$$\therefore\ MU = OV = AO; \quad \therefore\ PU \text{ is a tangent.}$$

4. *If an equilateral hyperbola be described, having center* O *and* OA *the semi transverse axis, the ordinate of the hyperbola is equal to the arc of the catenary.*

For, let $AR$ be the hyperbola,

$$\text{then, } VN^2 = (NO + OA)\,AN = RN^2;$$

$$\therefore\ RN = VN = PU = AP.$$

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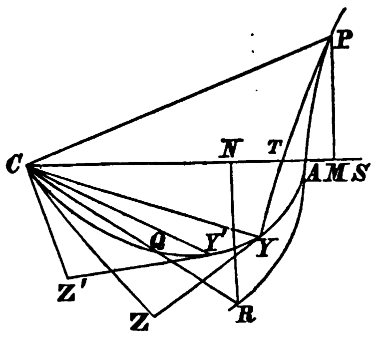
## LEMNISCATE.

Def. The lemniscate is the locus of the feet of the perpendiculars drawn from the center of a rectangular hyperbola upon the tangent.

1. *To find the inclination of the radius from the center of the lemniscate to the tangent at any point.*




Let $CY$ be perpendicular on $PT$ the tangent at the point $P$ in the hyperbola.

$$CY = PF;$$

$\therefore\ CY.CP = PF.CD = AC^2$,

$\because\ CP = CD$ in the rectangular hyperbola.

Draw the ordinate $PM$,

$$CT.CM = AC^2 = CY.CP;$$

$$\therefore\ CY : CT :: CM : CP;$$

and $CMP$, $CYP$ are right angles; $\therefore\ \angle PCM = \angle ACY$.

Draw $CZ$ perpendicular on the tangent at $Y$ to the lemniscate.

$\therefore\ ZCY$ and $YCP$ are similar triangles, (page 58, 3);

$\therefore\ \angle ZYC = \angle CPY =$ complement of twice $\angle YCA$.

2. *To find the perpendicular on the tangent at any point of the lemniscate.*

$$CZ \cdot CP = CY^2, \text{ and } CY \cdot CP = AC^2;$$
$$\therefore CZ : CY :: CY^2 : AC^2;$$
$$\therefore CZ \cdot AC^2 = CY^3.$$

3. *To find the chord of curvature through the center.*

Let $YV$ be the chord of curvature;

$\therefore YV : 2CZ :: CY - CY' : CZ - CZ'$, ultimately, (Art. 80),

$$\text{and } (CZ - CZ')\, AC^2 = CY^3 - CY'^3;$$
$$\therefore CY - CY' : CZ - CZ' :: AC^2 : 3CY^2;$$
$$\therefore YV : 2CZ :: CY : 3CZ;$$
$$\therefore YV = \tfrac{2}{3} CY.$$

4. *To find the radius of curvature.*

The radius of curvature

$$= \tfrac{1}{2} YV \cdot \frac{CY}{CZ} = \frac{CY^2}{3CZ} = \frac{AC^2}{3CY} = \tfrac{1}{3} CP$$

$= \frac{1}{3}$ of the radius of curvature at the corresponding point of the hyperbola.

5. *To find the area of the lemniscate.*

The sectorial area $ACQ$ may be shewn by Lemma IV. to be equal to the triangle $CRN$ where $CQ$ meets the auxiliary circle in $R$ and $RN$ is perpendicular to $CA$.

6. *To find the law of force tending to the center, under the action of which the lemniscate is described.*

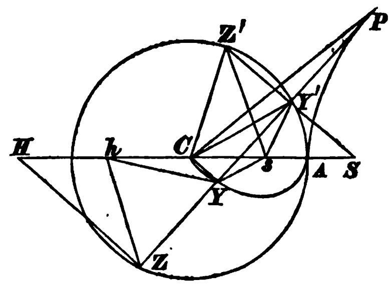
$$F = \frac{2h^2}{CZ^2 \cdot YV} = \frac{3h^2}{CZ^2 \cdot CY} = \frac{3h^2 AC^4}{CY^7} \propto \frac{1}{CY^7}.$$

7. *The velocity varies inversely as the cube of the distance.*

8. *To find the time in any arc of the lemniscate.*

$$\text{Time in } AQ = \frac{CN \,.\, NR}{h} = \sqrt{3} \,.\, \frac{AC^2 . CN \,.\, NR}{\mu^{\frac{1}{2}}}.$$

9. *To find the poles of the lemniscate.*




Let $S$, $H$ be the foci of the hyperbola,

$s$, $h$ the middle points of $CS$ and $CH$.

Draw $SY'Z'$ perpendicular to the tangent, meeting the auxiliary circle in $Y'$, $Z'$.

Join $sY'$, $sZ'$, $sY$, $hY$;

$\because$ $Cs = sS$, $sY' = sY$, and similarly $hY = hZ = sZ'$.

The altitude of the triangle $Y'CZ'$ is double that of $Y'sZ'$, upon the same base;

$$\therefore\ \Delta Y'CZ' = 2 \,.\, \Delta Y'sZ',$$

$$\text{and } CS \,.\, Ss = \tfrac{1}{2} CS^2 = AC^2 = SY' \,.\, SZ';$$

$\therefore$ a circle may be drawn circumscribing $CsY'Z'$;

$$\therefore\ \angle\, Y'CZ' = \angle\, Y'sZ';$$

$\therefore$ Since the triangle $Y'CZ'$ is double of $sY'C$,

$$sY' \,.\, sZ' = \tfrac{1}{2}\, CY' \,.\, CZ' = \tfrac{1}{2}\, CA^2;$$

$$\therefore\ sY \,.\, hZ = \tfrac{1}{2}\, CA^2,$$

which is the property of the poles of the lemniscate.

For this proof I am obliged to Professor Tait.

# GENERAL PROBLEMS.

## XIX.

1. Find the limit of $\dfrac{1.2 + 2.3 + \dots + n(n+1)}{n^3}$, when $n$ is indefinitely increased.

2. Prove, without finding the actual values, that the chords of curvature through the focus and center, and the diameter of curvature at any point of an ellipse, are as $\dfrac{1}{AC} : \dfrac{1}{CP} : \dfrac{1}{PF}$.

How does it appear that the chords of curvature through the two foci are equal?

3. A body describes an ellipse about one focus; prove that it always moves as fast *towards* one focus as *from* the other.

4. A particle describes a parabola round a force in the focus. $A$ is the vertex, $L$ the extremity of the latus rectum, $P$ a point whose distance from the axis is the length of the latus rectum. Prove that the time in $AL$ : time in $LP$ :: 2 : 5.

5. A body is revolving in an ellipse round a force in the center, and when the body arrives at the extremity of the major-axis, the center of force is transferred to the further focus; shew that the (eccentricity)$^2$ of the new orbit $= \dfrac{2e}{1+e}$, $e$ being that of the old orbit.

6. A body perfectly elastic, revolving in an ellipse about the focus, strikes a hard plane; if $\phi$, $\theta$ be the angles which the direction of its motion makes respectively with the focal distance and the plane, shew that the periodic time will be

unaffected, and that the new minor-axis will equal the former minor-axis $\times \dfrac{\sin(\phi + 2\theta)}{\sin\phi}$.

7. In question 6, find what would be the eccentricity of the new orbit if the old orbit were a circle. And if the old orbit were a parabola find what would be the inclination of the axis of the new orbit to the axis of the old one.

8. A balloon was found to be sailing steadily before the wind at an invariable elevation above the earth. A seconds pendulum suspended in the car was observed in 50 minutes to make 2997 oscillations; at what height was the balloon?

(radius of earth = 4000 miles, nearly.)

9. Shew how to find the weights of equal bodies on planets which have secondaries.

## XX.

1. If the sides of a right-angled triangle vary, while its area remains constant, determine the ultimate ratio of the changes in the sides adjacent to the right angle. Also shew that in the limit the increment of the hypothenuse is to the increment of the sum of the other two sides as the sum of these sides is to the hypothenuse.

2. The curvatures at the extremities of the major and minor axes of an ellipse are as 8 to 1; find the eccentricity.

3. If a particle describe an ellipse under the action of a force tending to the focus, and $v$, $v'$ be the velocities at two points equally distant from the axis on the same side, $V$ the velocity at the extremity of the minor-axis; prove that $vv' = V^2$.

4. Shew that, an ellipse being described under the action of a force tending in a direction perpendicular to the major-axis, the velocity varies as the secant of the angle which the direction of motion makes with the major-axis.

5. A body describes a circle about a force $F$ tending to the center $C$, shew that the arc $ADE = 2AC$ is described in the same time as a body would describe the chord $AE$ under the action of the force $F$ acting constantly parallel to $AC$.

6. A hyperbola and its conjugate are described by particles round a force in the center. They are at an apse at the same instant; shew that they will always be at the extremities of conjugate diameters. Also if $v$, $v'$ be their velocities,

$$v^2 - v'^2 = \mu (a^2 - b^2).$$

7. A body is projected with a velocity equal to that in a circle at the same distance at an angle of 30°, and acted on by a central force varying as the distance; determine the position, form, and magnitude of the orbit.

8. When force $\propto$ (dist.)$^{-2}$, shew that however the absolute force ($\mu$) be altered so that similar ellipses are described, the alterations of the absolute force and mean distance $a$ are in the ratio of $\mu : a$.

9. Find the time of oscillation in a cycloid; and the height of a mountain to the top of which if a seconds pendulum be carried, 43 oscillations are lost in a day; prove that it is about two miles high.

10. Shew that in the elliptic orbit described under the action of a force tending to a focus, the angular velocity round the other focus varies inversely as the square of the diameter parallel to the direction of motion.

## XXI.

1. $AB$ is an arc of finite curvature in any curve; the tangents at $A$ and $B$ intersect each other in $T$; and around the triangle $ABT$ a circle is described; when $B$ moves up to $A$, this circle ultimately bisects the diameter of curvature and all the chords of curvature.

2. Deduce the expression for the diameter of curvature at any point of a plane curve from the definition, that the circle of curvature is the limiting position of the circle passing through three consecutive points of a curve.

3. A body describes an ellipse about one focus; prove that (vel. at extremity of one latus rectum)×(vel. at extremity of other latus rectum) = (vel. at extremity of axis-minor)$^2$.

4. If the eccentricity of an ellipse be $\frac{1}{2}$, the time of moving under the action of a force tending to the center from one extremity of the latus rectum to the other is $\frac{\pi}{3\sqrt{\mu}}(3 \pm 1)$.

5. Given the velocity and direction at two points of a central orbit, find the locus of the center of force.

6. A body describes an ellipse about a center of force in the center; prove that if $r$, $r'$ be two radii vectores and $\alpha$ the angle between them, the time of describing the intercepted arc

$$= \frac{1}{\sqrt{\mu}} \sin^{-1}\left(\frac{rr' \sin \alpha}{ab}\right).$$

What is this time when $rr' \sin \alpha = \frac{1}{2} ab$, and the periodic time in the ellipse = 12 days?

7. If a closed string, lying on a smooth horizontal plane, pass loosely round three vertical pegs in the angles of an equilateral triangle, and if a bead be projected along the string so as to keep it stretched tightly, shew that the tension of the string will have two minimum values, and that they will be inversely proportional to the free lengths of the string in the two cases.

8. If the earth's orbit be taken an exact circle, and a comet be supposed to describe round the sun a parabolic orbit in the plane of the ecliptic; shew that this comet cannot possibly continue within the earth's orbit longer than the $\left(\frac{3\pi}{2}\right)^{\text{th}}$ part of a year.

9. A body describes a hyperbola, under a repulsive force tending from the farther focus, and when the body arrives at the vertex, the force suddenly becomes attractive; shew that, if the new orbit be a parabola, $e'$ the eccentricity of the hyperbola $= 3$; if the new orbit be an ellipse of eccentricity $e$, $e' \dot{-} e = 2$.

10. A particle slides down the arc of a vertical circle, starting from rest at a given point; find the point where it will leave the curve.

## XXII.

1. Find the ultimate ratio of the area of a segment of a circle to the area of a triangle on the same base, and whose vertex divides the arc in a given ratio when the arc is diminished without limit.

2. From a point in the circumference of a vertical circle a chord and tangent are drawn, the one terminating at the lowest point and the other in the vertical diameter produced; compare the velocities acquired by a heavy body in falling down the chord and tangent when they are indefinitely diminished.

3. A flat ring is revolving about its center with a given angular velocity; find the law of force under the action of which it would continue to revolve exactly as before if cohesion were destroyed.

4. A hollow cylinder consists of particles attracting as the distance; shew that if a particle be projected along the interior, with any velocity, in a plane perpendicular to the axis, it will continue to make isochronous oscillations between points at equal distances above and below the middle section.

5. If a body be projected with a velocity $= \sqrt{2} \times$ velocity in a circle at the same distance at an angle of $45^0$, determine the orbit completely. Force $\propto$ (dist.)$^{-2}$.

6. Supposing the major-axis of an ellipse = 200 feet, the eccentricity $= \frac{1}{10}$, and the periodic time 10 days; find the number of square inches in the area swept out by the radius vector in $1''$.

7. Two parabolas have the same axis, and the vertex of one of them lies half-way between the focus and vertex of the other, which is intersected by the first at the extremities $LL'$ of the latus rectum of the other. Particles being in motion in these parabolas under the action of forces to the respective foci, compare the times of moving from $L$ to $L'$.

8. A particle describes an ellipse, the center of force being situated at any point within the figure. Shew that at the point where the *true* angular velocity is equal to the *mean* angular velocity, the radius vector is a mean proportional between the semiaxes.

9. A body describes a circle to the center of which it is connected by a string; it is attracted to a point in the circumference by a force varying as the distance; shew that if the string be always kept stretched, the greatest and least velocities are in a ratio less than $\sqrt{3} : 1$.

10. A particle moves from any point in the directrix of a conic section, in a straight line towards a center of force $\propto \frac{1}{(\text{dist.})^2}$ in the nearer focus. Prove that, when it arrives at the conic section, its velocity $= \sqrt{\frac{\mu}{\frac{1}{4}\text{ latus rectum}}}$.

## XXIII.

1. *ACB* is an arc of a curve of continued curvature; find the ultimate ratio of the area of the triangle formed by joining the points *A*, *B*, *C*, to that of the triangle included between the tangents at those points.

2. Apply Lemma IV. to prove that the area included between a hyperbola and the tangents at the vertices of the conjugate hyperbola is equal to the area included between the conjugate hyperbola and the tangents at the vertices of the hyperbola.

3. A body describes an ellipse about a center of force in the focus $S$, and $PP'$ is a chord parallel to the major-axis; shew that the time of describing the arc $PP'$ : time of describing the whole ellipse :: $\sin^{-1}.\dfrac{AC-SP}{SC} : \pi$.

4. Having given rad. of earth = 4000 miles nearly, shew that gravity in latitude $\lambda = G\left(1 - \dfrac{\cos^2\lambda}{289}\right)$, the earth being considered spherical, and $G$ gravity at the pole.

5. The sides $a$, $b$, $c$ of a triangle are composed of matter attracting directly as the distance, with an intensity which would equal $\mu x$ at the distance $x$ if the whole matter were collected at a point: from $D$, $E$, $F$, the middle points of the sides, three particles are projected in the directions $DE$, $EF$, $FD$, with velocities $\sqrt{\mu c}$, $\sqrt{\mu a}$, $\sqrt{\mu b}$: now if $S$ be the sum of the areas of the three orbits, and $A$ be the area of the triangle, shew that $S = \pi A$.

6. A particle is attached by an elastic string to a center of force of constant intensity, and of such magnitude that it would exactly double the length of the elastic string. The string is now stretched and the particle projected at right angles to it. Shew that the particle will begin to move in an ellipse; but if the velocity of projection be less than the velocity in a circle at the same distance, the ellipse will be deserted after a certain interval of time.

In the latter case find the velocity and direction of motion at the moment of leaving the ellipse.

7. The latus rectum of a comet's parabolic orbit is equal to the diameter of the earth's orbit supposed circular; if the earth describe an arc of its orbit equal to the radius in $58\frac{1}{2}$ days, find how long the comet takes to move from one extremity of the latus rectum to the other.

8. Shew that if a body describe an ellipse of very small eccentricity under the action of a force tending to a focus, the angular velocity about the other focus is very nearly uniform.

9. If any number of bodies describe parabolas about a common center of force in the focus, the square of the time of passing from one extremity of the latus rectum to the other varies as the cube of the latus rectum.

10. The angular velocity of a body describing an ellipse whose eccentricity $= \dfrac{1}{\sqrt{2}}$ is $10''$ an hour at the extremity of the latus rectum; find the periodic time.

11. A perfectly elastic body falls from rest towards a center of force varying inversely as the square of the distance, and when it has fallen half the distance it is reflected by a plane so as to move in a direction making an angle $\alpha$ with its former direction; shew that the eccentricity of the ellipse described is $\cos\alpha$.

12. Shew that the intersection of the string of a cycloidal pendulum, which makes complete oscillations with the base of the cycloid, moves uniformly along the latter.

## XXIV.

1. $AQ$ is an arc of continued curvature, $AR$ a tangent at $A$, and of length equal $AQ$; shew that $QR$ is ultimately perpendicular to the tangent.

2. If $AB$, $A'B'$, two chords of a curve of equal length, cut each other in $T$, shew that if $A'B'$ approach to and coincide with $aB$, then $AT : BT = \tan\alpha : \tan\beta$, ultimately, where $\alpha$, $\beta$, are the angles that $AB$ makes with the tangents at $A$ and $B$.

3. A constant force $f$ acts upon a particle $P$ in the direction $SP$, $P$ is also at the same time attracted towards $S$ with a force $\frac{\mu}{SP^2}$; shew that if $a$ be the distance $PS$ at the commencement of motion from rest under these forces, the particle will move off to *infinity*, or not, according as $a$ is $>$ or $< \mu f^{-\frac{1}{2}}$.

4. When a body describes a parabola about the focus, the intersection of its direction with the axis of the parabola moves most rapidly when the body is at the extremity of the latus rectum.

5. A particle is describing an ellipse round a force in the center, and when it arrives at the extremity of the minor-axis, the force is replaced by one which is equal in magnitude at that point, but varies inversely as the square of the distance; shew that the eccentricity of the new orbit is $\frac{CS^2}{BC^2}$.

6. A body moves in elliptic arcs about a center of force varying as $\frac{1}{(\text{dist.})^2}$ situated in a perfectly elastic plane perpendicular to the plane of the orbits; shew that those arcs are portions of similar ellipses whose major-axes are equally inclined to the elastic plane, and that the time between the first and third impact is equal to that between the second and fourth.

7. The angular velocities of a body moving in an ellipse about a force in the center are $4^0$ and $9^0$ per hour at the extremities of the major and minor-axes respectively; find the periodic time.

8. Sir John Herschel states that the great comet of 1843

passed within a distance equal to $\frac{1}{7}$th of the sun's radius from the sun's surface. Taking the sun's diameter as 882,000 miles, and the earth's distance from the sun as 95,000,000 miles, find the velocity of the comet at perihelion.

9. *AB* is the vertical axis of a cycloid, *A* the highest point, *AM*, *AN* are the abscissæ of points at which a body begins to slide down the arc of the cycloid, and at which it leaves the curve; prove that *N* is the middle point of *MB*.

10. A particle moves in a smooth elliptic tube, at the foci of which are situated two centers of force of unequal intensity, the one attracting and the other repelling, according to the law of the inverse square; find the pressure. Shew that there exists a certain circle, such that a particle placed anywhere on its circumference, and abandoned to the free action of the forces, will describe an ellipse having those centers of force for the foci.

11. A body acted on by a central force which varies inversely as the square of the distance, is constrained to move in a circle whose radius is $a$, and center is at distance $b$ from the center of force; it is projected with velocity $V$ from the nearer extremity of the diameter which passes through the center of force; shew that, in order that it may complete the circuit,

$$V^2 \text{ must at least } = \frac{2\mu}{a-b} - \frac{2\mu}{a+b}.$$

## XXV.

1. In a curve are placed two chords *AB*, *AC*; and *BC* is parallel to the tangent at *A*; prove that when *B*, *C* move up to *A*, *AB* and *AC* are ultimately in a ratio of equality.

2. If a line move parallel to the base of a cycloid, find the limiting ratio of the segment of the cycloid to the corresponding segment of the generating circle, as the line becomes infinitely near to the vertex.

3. If $v$ and $v'$ be the velocities at extremities of chord $PSP'$ in parabola $\frac{v}{v'} = \cot \frac{ASP}{2}$.

4. A heavy particle is projected horizontally from any point in the interior of a surface of revolution, whose axis is vertical; the velocity being that due to the height, above a given horizontal plane, of the point of projection, find the form of the surface so that the particle may always remain in the horizontal plane of projection.

5. A planet and a comet moving in the same plane have the same periodic time; shew that if the planet's orbit be a circle the ratio of the time during which the comet is beyond the orbit of the planet : the time during which it is within the same must be $> 1$ and $< 4.50388$.

6. The velocity generated by gravity in a free body at the earth's surface being 32 feet per second, the earth's radius being 4000 miles, and the moon's mean distance from the earth $60 \times$ the earth's radius, find approximately the number of days in the moon's period.

7. Different bodies are projected about a center of force $\propto (\text{dist.})^{-2}$ with the velocity acquired in falling from an infinite distance to the common point of projection; shew that the envelope of all the orbits is a sphere.

8. A body falls from a distance and towards a center of force $C$, force varying as the distance. When it has described a space $\frac{1}{2} a$ it impinges at an angle of $45^\circ$ on a perfectly elastic plane and is reflected. Shew that the semiaxis of the orbit subsequently described will be $a \cos \frac{1}{3}\pi$ and $a \sin \frac{1}{3}\pi$. Suppose that the body again impinges on the same fixed reflecting plane, shew that it will be reflected to the center, and that time of arriving at the center : time of falling directly to it :: 3 : 1.

9. Suppose $e$ to be the elasticity of the plane in the last problem, prove that, if the angle of incidence $= \tan^{-1}\sqrt{e}$, the subsequent orbit will have its axis major or minor in the direction in which the body was originally falling, according as the distance from the center $C$ to the point of impact is

$$> \text{ or } < a\sqrt{\frac{e}{1+e}}.$$

10. A body revolves in an ellipse about the focus; its angular velocities at the further and nearer apse are respectively $1^\circ$ and $16^\circ$ per hour; find the eccentricity of the ellipse and the periodic time.

11. A particle describes an ellipse round a force in one focus; at what point of the orbit may a given *finite* change be made in the *direction* of the motion without changing the position of the apse line?

12. A body describes a parabola about the focus; if the segments $PS$, $Sp$ of the focal chord $PSp$ be in the ratio $n : 1$, prove that the time of describing $pAP$

$$\text{is } \frac{2}{3}\frac{(AS)^{\frac{3}{2}}}{\sqrt{2\mu}}\left(\sqrt{n}+\frac{1}{\sqrt{n}}\right)^3.$$

13. If $P$ be a point in a cycloid and $O$ the corresponding position of the center of the generating circle, shew that $PO$ touches another cycloid of half the dimensions.

## XXVI.

1. $ABC$ is a right-angled triangle, whose hypothenuse is constant; prove that if $AB$, $ab$ be two portions of the hypothenuse, intersecting at $P$,

triangle $PAa$ : triangle $PBb$ :: $CB^4$ : $CA^4$, ultimately.

2. Apply the fourth Lemma to shew that the attraction of a point $S$, by a uniform straight rod $AB$ is the same as that

of a circular arc of the same substance whose center is $S$ and which touches $AB$ and has its bounding radii in $SA$ and $SB$.

3. If $PQ$ be an arc of a plane orbit described by a body in time $T$, and any two lines $Pm$, $Qm$ be drawn at right angles to each other, then the measure of the accelerating effect of the forces at $P$ parallel to these two lines are respectively

$$2 \times \text{limit } \frac{Pm}{T^2}\left(1 - \frac{V.T}{PQ}\right), \text{ and } 2 \times \text{limit } \frac{Qm}{T^2}\left(1 - \frac{V.T}{PQ}\right)$$

when $PQ$ and $T$ are diminished indefinitely, $V$ being the velocity at $P$.

4. An ellipse and a hyperbola have the same center and foci. They are described by particles, under the action of forces in the center of equal intensity. If $a$, $a'$ be their semi-transverse axes, the square of the velocity of *each* body at a point where the curves cut $= \mu (a^2 - a'^2)$.

5. A particle describes an ellipse about a center of force in the focus, and another particle describes the circle upon the axis-major about another force in the same point in the same periodic time. If the particles start simultaneously from the vertex, prove that the line joining them is always perpendicular to the axis.

Also shew that the velocity at any point in the circle is inversely proportional to the corresponding focal distance in the ellipse.

6. Find the law of force tending to the pole of the cardioid.

7. Bodies describing ellipses about a given center of force which $\propto (\text{dist.})^{-2}$ pass through a given point with the velocity in a circle at that distance: the locus of the vertices of the ellipses is the cardioid, the center of force being pole.

8. Two particles move in different planes about a center which attracts with a force varying inversely as the square of the distance, the one in a circle, the other in an ellipse; the orbits have two points in common, and at either of these points the velocity of one particle is to that of the other as $n$ to 1. Determine the eccentricity of the ellipse.

9. A perfectly elastic particle is moving in a circle, under the action of a force to the center varying as $(\text{dist.})^{-2}$, and impinges on a plane perpendicular to the plane of motion and making an angle of $60^\circ$ with the radius from the point of impact. Find the new orbit and the time during which the particle is within its former orbit.

10. Find the axis of the new parabola described if the body, supposed perfectly elastic, strike a hard plane interposed at any point in its path.

11. Define angular velocity and mean angular velocity: if $u$, $v$ be the angular velocities at the extremities of the major-axis of an ellipse, the center of force being in the focus, find the angular velocity at the extremity of the minor-axis.

12. An ellipse is described about a center of force in the focus. A parabola is described with its axis coincident in direction with the minor-axis, so as to pass through the points $X$, $X'$, where the axis-major produced meets the directrices, the latus rectum being $2ae^{-2}$. If we draw any line parallel to the axis-minor cutting the ellipse in $P$, the parabola in $Q$, and the axis-major in $N$, then will $QN$ be the space due to the velocity at the point $P$.

13. A body is attached to the end of a string, which just winds round the circumference of a circle, in whose center there is a repulsive force $= \mu$ (dist.). Prove that the time of unwinding $= \dfrac{2\pi}{\sqrt{\mu}}$. Also, find the tension of the string at any time.

## XXVII.

1. Shew that the limit of the whole length of the hypocycloid or epicycloid corresponding to a complete revolution of the generating circle is eight times the radius of the fundamental circle, when that of the generating circle is indefinitely diminished.

2. If $\alpha$, $\beta$ be the greatest and least angular velocities in an ellipse described about the focus, the mean angular velocity is equal to $\dfrac{2\sqrt[4]{\alpha^3\beta^3}}{\sqrt{\alpha}+\sqrt{\beta}}$.

3. A particle describes with uniform velocity an equiangular spiral whose constant angle is 45°; shew that its motion may result from the attraction of a center of force varying as $\dfrac{1}{\text{dist.}}$, which itself moves with the same uniform velocity in a certain other similar and equal spiral.

4. A body is describing a circle with a uniform velocity $v$, and in a periodic time $P$; if a velocity $nv$ be communicated to it in direction of the center, shew that a diameter of the circle is the latus rectum of the new orbit, and the periodic time in the new orbit is $= P(1-n^2)^{-\frac{3}{2}}$.

5. Two bodies describing the same ellipse about the same center of force in the focus start together from the two extremities of the major-axis. The angles which they have described will have the greatest difference, when the area included between their distances from the focus is half the area of the ellipse.

6. The velocity of a body describing a hyperbola by the action of a repulsive force in the center is at any point the same as if it had been repelled to that point in a straight line from rest when at a distance from the center $= \sqrt{a^2 - b^2}$.

7. A given quantity of matter, consisting of particles which attract with forces varying as the distance, is formed into a thin hemispherical shell. Shew that, whatever be the size of the hemisphere, a particle placed at a given angular distance from the vertex will always reach that point in the same time.

8. A particle moves in an elliptic tube under the attraction of a material line joining the foci, each element of which attracts with a force varying inversely as the square of the distance. Shew that the velocity is constant; and find the pressure on the tube when the particle is at the extremity of the minor-axis.

## XXVIII.

1. If any number of particles be moving in an ellipse about a force in the center, and the force suddenly cease to act, shew that after the lapse of $\left(\frac{1}{2\pi}\right)^{\text{th}}$ part of the period of a complete revolution all the particles will be in a similar, concentric, and similarly situated ellipse.

2. Find by Newton's method the law of force parallel to one of the asymptotes, by which a body may describe an equiangular hyperbola.

3. If a particle in a smooth elliptic groove, under the action of two centers of force in the foci, each varying inversely as the square of the distance, the absolute forces being the same, be placed at the extremity of the axis-minor, prove that the equilibrium will be unstable; but if at the extremity of the axis-major it will be stable, and in this latter case, shew that the time of a small oscillation is $\pi\left(\frac{b^2}{a}\right)^{\frac{3}{2}}\div 2e\sqrt{\mu}$.

4. Two bodies of equal mass and whose coefficient of elasticity is $\frac{1}{2}$, are revolving in the same ellipse (eccentricity $= \frac{3}{5}$), but in opposite directions round a center of force in the focus: they impinge upon one another at the nearest apse: determine the distances at which they will afterwards impinge on each other: and shew that the whole time from the first impact to their falling into the center of force is $\frac{\pi}{14} \cdot \frac{(5p)^{\frac{3}{2}}}{\sqrt{2\mu}}$, where $p$ is the least distance at first, and $\mu$ the absolute force.

5. A body is projected about a center of force $\propto (\text{dist.})^{-2}$ perpendicularly to the distance: shew that as the velocity of projection is increased the center of the curve moves through the center of force to infinity, it then suddenly starts back to the other side of the point of projection and goes off to infinity in that direction. But when the force $\propto$ dist. the nearer focus moves to a given point and then suddenly starts at right angles to its previous direction.

6. Two perfectly elastic balls are moving in concentric circular tubes in opposite directions and with velocities proportional to the radii: at an instant when they are in the same diameter and on opposite sides of the center the tubes are removed and the balls move in ellipses under the action of a force of attraction in the common center of the circles varying inversely as the square of the distance. After one has performed in its orbit a complete revolution and the other a revolution and a half, a direct collision takes place between the balls and they interchange orbits; find the relation between the radii of the circles and between the masses of the balls.

7. A body describes an ellipse in a free medium under the attraction of two equal forces, one in each focus, varying at any point as $\frac{1}{c^2}$, $c$ being the semiconjugate diameter at that point: if the medium were to resist with a force varying as any

function of the velocity, the body might be made to describe the same ellipse in the same manner by increasing the force in one focus and diminishing that in the other by a quantity which varies as $\frac{c}{\sqrt{c^2 - b^2}}$, $b$ being the semiaxis-minor.

8. An attractive force equal to $\frac{\mu}{(\text{dist.})^2}$ resides in each focus of a smooth elliptic groove; if a particle start from the end of the major-axis with a velocity $\frac{2\sqrt{\mu a}}{b}$, it will reach the end of the minor-axis in a time $\frac{\pi a^{\frac{3}{2}}}{4\sqrt{\mu}}\left(1 - \frac{e^2}{2}\right)$, $a$, $b$, $e$ being the semiaxes and eccentricity.

## RESULTS OF PROBLEMS.

### I.

1. In (1) and (3) the limit is $a$, in (2) the ratio decreases indefinitely.

2. (1) 3 (2) $\frac{1}{3}$. 3. $a : b$.

14. $1+1+\dfrac{1}{1.2}+\dfrac{1}{1.2.3}+\dfrac{1}{1.2.3.4}+\ldots\ldots$ *ad inf.*

15. 1.

### II.

6. The distance from the base is $\frac{1}{4}$th of the height.

9. The mass is $\dfrac{2}{m+2}$ of the homogeneous circle whose density is equal to that of the given circle at the circumference.

10. $\dfrac{\pi b^4}{4a}$ is the volume generated by the closed portion.

### III.

5. $\frac{1}{3}$rd of the mass of an equal homogeneous rod of density the same as at the extremity of the given one.

6. $\frac{1}{4}$th of the cylinder of equal height on the same base.

### IV.

7. The angle between the curve and radius must be the same in both.

### VII.

2. Where the curve is made to pass through $A$.

6. The curve is a parabola passing through $A$, vertex at a distance $a$ below $AK$, and its axis at an unit of distance from $A$ meeting $KA$ produced, latus rectum is $a^{-1}$, space is $\frac{1}{3}a(3t^2+t^3)$.

7. The relation between the abscissa and ordinate is the same as between the arc and sine of the arc of a circle.

8. $(\text{vel.})^2$ at $M = \mu(SM^2 - SA^2)$.

10. $\dfrac{\pi}{\sqrt{\mu + \mu'}}$.

## VIII.

6. $AD$ passes through the center of curvature ultimately.

## IX.

1. The fixed point is in $UA$ produced at a distance from $A$ equal to $AU$.

2. If through the point of the circle a chord of a quadrant be drawn, the directrix is parallel to this chord, the focus in the point of quadrisection, and the latus rectum is half the chord.

3. The focus of the parabola is in the base of the cycloid.

4. $1 : 2$. 5. See Art. 88.

## X.

2. The direction is not changed, but the curvature is changed in the ratio of new force to the old.

3. At the extremities of the minor and major axes.

10. $8\pi + 3\sqrt{3} : 4\pi - 3\sqrt{3}$. 11. $\dfrac{1-m}{1+m}\cdot\dfrac{\pi}{2}$.

12. See page 64. 16. Employ the auxiliary circle.

## XI.

2. $\left(\dfrac{9a^2P^2}{4\pi^2}\right)^{\frac{1}{3}}$. 3. areas $\propto$ (dist.)$^{-\frac{1}{2}}$. 4. $108\pi^2 : g$.

6. The day is shortened in the ratio of 17 : 1 nearly.

7. $V^2 = 3FR$.

## XIII.

1. $(\mu g l)^{\frac{1}{2}}$.

3. The velocity generated by the impulse $= \left(\dfrac{Wgl}{W'}\right)\omega - l$, ($\omega$ being the angular velocity before the impulse).

4. $\text{Cos}^{-1}\left(\dfrac{3}{2} - \dfrac{W}{2W'}\right)$.

5. $AM$, $AN$ abscissæ of point from which the body falls and the point at which the pressure is required, $NQ$ ordinate of generating circle; pressure : weight :: $BN + MN : BQ$.

7. $2a$ = length, $b$ = greatest depth,

$$\text{tension} : \text{weight} :: a^2 + 2b^2 : 2ab.$$

9. See Art. 119.

10. Normal force whose accelerating effect $= \dfrac{V^2 \sin a}{SP}$.

## XIV.

1. $P_{\prime}^2 . S_{\prime}P^2 . PV_{\prime}^3 : P^2 . SP^2 . PV^3$.

3. The unit of time is that which is employed in fixing the measure of the accelerating effect of the force.

## XVII.

2. Let the change take place at $A$, $C$ the center.

(1) The semiaxes are $CA$ and $\frac{1}{\sqrt{n}} . CA$.

(2) The semiaxes are $CA$ and $n . CA$.

(3) The orbit is a rectangular hyperbola, vertex at $A$.

(4) The axes are $(\sqrt{5}+1)\,CA$, and $(\sqrt{5}-1)\,CA$, and the inclination of the major-axis to $CA = \frac{1}{2}\sec^{-1}\sqrt{5}$.

4. The new axes are $CA$ and $2CA$.

5. If $BC$ be the semiminor-axis, this is one of the axes of each orbit, and the others are $\frac{m - em'}{m + m'} . AC$, $\frac{m(1+e)}{m+m'} . AC$.

6. If $\mu m D$ be the measure of the accelerating effect of the attraction of a mass $m$ at a distance $D$, time $= \frac{2\pi}{\sqrt{\mu m}}$.

11. The point of projection corresponding to the greatest ellipse is the point of bisection of $CA$.

## XVIII.

1. $\frac{1}{2}$. 5. $c$ = perihelion distance of comet, $a$ = radius of earth's orbit, time $\propto (a+2c)(a-c)^{\frac{1}{2}} = \sqrt{2}\,a^{\frac{3}{2}}$, if $2c = a$, and $2a^3 - (a+2c)^2(a-c) = (a+c)(a-2c)^2 > 0$.

6. About 225 days. 7. About an hour.

8. The angles of the auxiliary circles corresponding to the point of intersection of the two orbits are $30^\circ$ and $60^\circ$, whence it is easily shewn that the ratio required is $\sqrt{3}(3+2\pi) : 8\pi - 9$.

9. The transverse axis is the semi latus rectum, and is perpendicular to the axis of the parabola, and eccentricity $= \sqrt{3}$.

10. The locus is a circle, whose center is the middle point of the distance.

11. The new orbit is a hyperbola, transverse axis = semi-major-axis of the ellipse, eccentricity = $\sqrt{9 - 8e^2}$. With the given relation the point at which the change takes place is the extremity of the latus rectum.

14. 39 days nearly. 16. semimajor-axis $= \dfrac{CA}{1 + e^2}$.

## XIX.

1. $\frac{1}{3}$. 7. $\sin 2\theta$. $4\theta$. 8. 4 miles 7 yards.

## XX.

1. The required ratio is that of the sides. 2. $\frac{1}{2}\sqrt{3}$.
7. $a = \frac{1}{2}(\sqrt{3} + 1)c$, $b = \frac{1}{2}(\sqrt{3} - 1)c$, $\varpi = 15^\circ$.

## XXI.

5. A straight line passing through the intersection of the tangents and making with them angles whose sines are inversely proportional to the velocities. 6. One day. 10. The body starting from a given distance from the horizontal diameter, leaves the curve at two-thirds of that distance.

## XXII.

1. $m : n$ the given ratio, the required ratio is $(m + n)^2 : 3mn$. 2. $1 : 2$. 3. $\omega^2 \times$ distance. 5. Latus rectum = twice the distance, and the axis is perpendicular to the distance. 6. $\frac{1}{2}\pi . \sqrt{11}$. 7. $2\sqrt{2} : 1$.

## XXIII.

1. $2 : 1$. 6. $c$ the initial, $a$ the natural length, $f$ the force, $u$ the velocity of projection in the latter case, $v$, $\alpha$ the required velocity and angle. $v^2 = \dfrac{f}{a}(c^2 - a^2) + u^2$. $\text{Sin}\,\alpha = \dfrac{cu}{av}$.

7. 78 days. 10. $10800\sqrt{2}$ days.

## XXIV.

1. Let $QT$ tangent at $Q$ meet $AR$ in $T$, on $AR$ measure $Aq$ equal to chord $AQ$ and $Tr = TQ$; now arc $AQ >$ chord $AQ$ and $< AT + TQ$, $\therefore$ $AR > Aq$ and $< Ar$, $\therefore$ $QR$ is between $Qq$, $Qr$ each of which is ultimately perpendicular to the tangent, $\therefore$ $QR$ is so. 7. 60 hours. 8. 366 miles per second. 10. In page 165, writing $-\mu'$ for $\mu'$, the reaction $R$ vanishes if $\mu HO^2 = \mu' SO^2$, $\therefore$ locus of $O$ is a circle.

## XXV.

2. $2 : 1$. 4. A paraboloid of revolution. 6. $\frac{5\pi}{6}\sqrt{110}$.

10. $\frac{3}{5}$. 112.5 hours. 11. If $\beta$ be the finite change, and $PM$ perpendicular to the axis, $HPM = \beta$, $HP = \dfrac{a(1-e^2)}{1 - e\sin\beta}$.

## XXVI.

6. Force $\propto$ (dist.)$^4$. 8. $e^2 = n^2 - 1$.

9. $a$ = radius, axis-major $= 2a$ and is inclined at $30^0$ to the radius, eccentricity $= \frac{1}{2}\sqrt{3}$, time required $= \dfrac{a^{\frac{3}{2}}}{\mu^{\frac{1}{2}}}(\pi - \sqrt{3})$.

10. The angle between the axes is four times that between the plane and tangent. 11. $\dfrac{4uv}{(\sqrt{u} + \sqrt{v})^2}$.

13. $T = m \,.\, 2a\mu^{\frac{1}{2}}t$.

## XXVII.

3. The path of the center of force is the evolute of the given spiral.

7. Shew that the accelerating effect of the force on the angular motion is the same at the same angular distance.

8. If $\frac{\mu}{D^4}$ be the accelerating effect of the attraction of an unit of mass collected in a point upon a body at a distance $D$, $n$ the number of units of mass in the material line, $W$ the weight of the body, $v$ its velocity, the pressure on the tube

$$= W \times \left(\frac{v^2 b}{a^2 g} \sim \frac{\mu n}{abg}\right).$$

## XXVIII.

1. The ellipse is the locus of the angular points of the circumscribing parallelogram whose sides are parallel to conjugate diameters: the semi-axes are $a\sqrt{2}$, $b\sqrt{2}$.

2. The force varies as the cube of the distance from the asymptote to which the force acts perpendicular.

3. If $\phi$ be the inclination of the normal at a point $P$ near $A$, shew that the force in the tangent has an accelerating effect $\frac{4\mu e a^2}{b^4} \cdot \sin\phi$, and $\phi = \frac{a \cdot AP}{b^2}$, then proceed as in Art. 8 of Appendix II.

4. Shew that at every impact the major-axis is diminished to $\frac{1}{4}$th, that the eccentricity is unaltered, and that the greatest distance in each orbit is the least in the preceding. The distances at which they impinge are $p$, $\frac{1}{4}p$, $\frac{1}{16}p$, $\frac{1}{64}p$, &c.

6. The masses are equal, and if $r$, $s$ be the radii of the circles $\frac{r^3}{s^3} - \frac{s}{r} = 1 - \left(\frac{2}{3}\right)^{\frac{3}{2}}$.

7. The motion being the same in both cases, the velocity in the resisting medium is constant, and therefore the resistance constant also. Hence, shew that $\frac{\mu}{c^2} \pm f$ being the two new focal forces, $f \cos\alpha$ is constant, and thence deduce the result.

8. If $PP'$ be an arc described in a small time, $PMQ$, $Q'P'M'$ common ordinates for the ellipse and auxiliary circle, $DN$ that of the extremity of the conjugate diameter to $CP$; shew that velocity at

$$P = \frac{2\sqrt{\mu a}}{CD}, \text{ or } \frac{2\sqrt{\mu a}}{CD} \cdot \frac{CN}{CD} = \frac{2\sqrt{\mu a}\, QM}{CD^2} \text{ parallel to } AC.$$

$$\therefore \text{ time in } PP' = \frac{MM'}{2\sqrt{\mu a}} \cdot \frac{CD^2}{QM} = \frac{MM'}{2\sqrt{\mu a}} \cdot \frac{e^2 QM^2 + b^2}{QM},$$

and making the summation from $A$ to $B$,

$$\Sigma\,(MM' \,.\, QM) = \frac{\pi a^2}{4}, \text{ and } \Sigma\left(\frac{MM'}{QM}\right) = \Sigma\,\frac{QQ'}{a} = \frac{\pi}{2}, \text{ ultimately};$$

$$\therefore \text{ time from } A \text{ to } B = \frac{1}{2\sqrt{\mu a}}\left(\frac{\pi a^2 e^2}{4} + b^2 \,.\, \frac{\pi}{2}\right) = \frac{\pi a^{\frac{3}{2}}}{4\sqrt{\mu}}\left(1 - \frac{e^2}{2}\right).$$

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